Financial Mathematics Preparation Course
Notes on Calculus

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1 The Real Numbers

The basis of calculus is the set of real numbers which we shall denote by the symbol $\mathbb{R}$. Loosely speaking the real numbers is the set of all infinite decimal expansions $a_0.a_1a_2a_3a_4\ldots$ where the a_i's are integers and a_i for $i \geq 1$ is between 0 and 9. Clearly all integers and all rational numbers $\frac{p}{q}$ where p and q are integers, $q \neq 0$ are real numbers (for instance the integer -124 can be written $-124.000000\ldots$, $\dfrac{1}{3} = 0.33333333\ldots$). We denote the set of integers $\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$ by the symbol $\mathbb{Z}$ and the set of all rational numbers by $\mathbb{Q}$. The set of positive integers $1, 2, 3, 4, 5, \ldots, n, \ldots$ is called the set of natural numbers and is denoted $\mathbb{N}$. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Both the natural numbers, the integers and rational numbers have the property of being *countable* i.e. we can write them all down in a sequence $z_1, z_2, z_3, \ldots, z_n, \ldots$. For instance we can write down all the integers in the sequence $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$. It is a little more tricky to see that the set of all rational numbers are countable but it can be done. The real numbers, however are not countable. Here is the argument: assume that $\mathbb{R}$ is countable and assume we have written down all the real numbers in a sequence:

$$\begin{array}{l}
a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}\dots \\
a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}\dots \\
a_{30}.a_{31}a_{32}a_{33}a_{34}a_{25}\dots \\
a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}\dots \\
a_{50}.a_{51}a_{52}a_{54}a_{54}a_{55}\dots \\
\vdots \\
\vdots
\end{array}$$

Thus the first index indicates the position in the sequence and the second the position in the decimal expansion. We shall show that this is impossible by constructing a real number which is not in this sequence. Choose $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33}, b_4 \neq a_{44}, b_5 \neq a_{55}, b_6 \neq a_{66}, \dots$. Now form the real number $0.b_1 b_2 b_4 b_4 b_5 b_6 b_7 b_8 \dots$. Assume this number is the n'th term in the sequence. Thus $0.b_1 b_2 b_3 b_4 \dots b_n \dots = a_{n0}.a_{n1}a_{n2}\dots a_{nn}\dots$, but since $b_n \neq a_{nn}$ they are not equal and so we have a contradiction. Since the rational numbers, $\mathbb{Q}$, is countable this shows that there are strictly more real numbers than rational numbers: $\mathbb{Q} \subsetneq \mathbb{R}$. Here is another argument, which goes all the way back to the ancient greeks, that shows that there are real numbers which are not rational: consider the real number $\sqrt{2}$. If this is a rational number we can write $\sqrt{2} = \frac{p}{q}$ where we assume that the fraction $\frac{p}{q}$ is in lowest terms i.e. the integers p and q have no common factors. By squaring both sides we get $2 = \frac{p^2}{q^2}$ or by cross-multiplying $2q^2 = p^2$. This show that p^2 is an even number, now if p itself was odd p would be of the form $2r+1$ for some integer r and so $p^2 = (2r+1)^2 = 2r^2 + 1 + 4r$ which is also odd. Hence in order for p^2 to be even p itself must be even so $p = 2r$. But then $p^2 = 4r^2$ and we get $2q^2 = 4r^2$. Dividing by 2 on both sides we get $q^2 = 2r^2$ so q^2 is even and hence also q must be even. This shows that 2 is a common factor in p and q in contradiction with the fact that $\frac{p}{q}$ is in lowest terms.

Real numbers can be added, multiplied and divided (if the number we divide by is $\neq 0$) just as rational numbers can. What sets the real numbers apart from the rational numbers, and which is the discovery that forms the foundation of calculus is the concept of a *limit*. This notion took some 2000 years to discover. The ancient greek mathematicians studied geometry and certainly discovered the number π as the ratio between the circumference of a circle and its diameter. They were, however incapable of computing

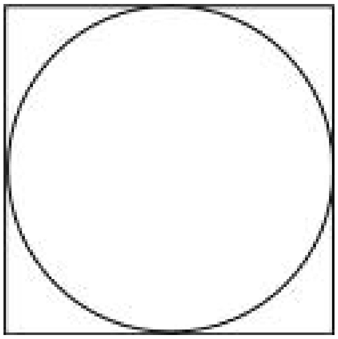

Figure 1:

π because as we now know π is not a rational number. The greeks were certainly comfortable with some non-rational numbers as they freely operated for instance with square roots but as we also know now π cannot be expressed even using square roots (or any other kind of roots, third, fourth, etc).

Example 1.1 *We want to compute the area of the unit disc i.e. the disc with radius* 1

Inscribing the disc in a square with sides $= 2$ *we get that the area* ≤ 4 *We divide the disc into* 4 *equal slices and divide the interval into two halves. We inscribe the quarter circle in the union of two rectangles as shown in fig.2. The height of the smaller of the two rectangles is* $\sqrt{1 - (\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$. *Thus the combined area of the two rectangles is* $\frac{1}{2} + \frac{\sqrt{3}}{4}$. *It follows that the area of the disc* $\leq 2 + \sqrt{3} \sim 3.7321$. *Next we divide the interval into quarters and draw four rectangles: the height of the rectangles are* $1, \sqrt{1 - (\frac{1}{4})^2}, \sqrt{1 - (\frac{2}{4})^2}, \sqrt{1 - (\frac{3}{4})^2}$. *The combined area is* $\frac{1}{4}(1 + \sqrt{1 - (\frac{1}{4})^2} + \sqrt{1 - (\frac{2}{4})^2} + \sqrt{1 - (\frac{3}{4})^2})$ *and so we get the estimate: area of the disc* $\leq 4(\frac{1}{4}(1 + \sqrt{1 - (\frac{1}{4})^2} + \sqrt{1 - (\frac{2}{4})^2} + \sqrt{1 - (\frac{3}{4})^2})) \sim 3.4957$.

In the next step we divide the intervals into $1/8$*'ths and we get the combined area of the eight rectangles is* $\frac{1}{8}(1 + \sqrt{1 - (\frac{1}{8})^2} + \sqrt{1 - (\frac{2}{8})^2} + \sqrt{1 - (\frac{3}{8})^2} + \sqrt{1 - (\frac{4}{8})^2} + \sqrt{1 - \frac{5}{8})^2} + \sqrt{1 - (\frac{6}{8})^2} + \sqrt{1 - (\frac{7}{8})^2})$. *From this we get the estimate: area of disc* ≤ 3.3398

We can continue this way, next we divide into $1/16$*'ths,* $1/32$*'ths,*$\ldots, 1/2^n, \ldots$.

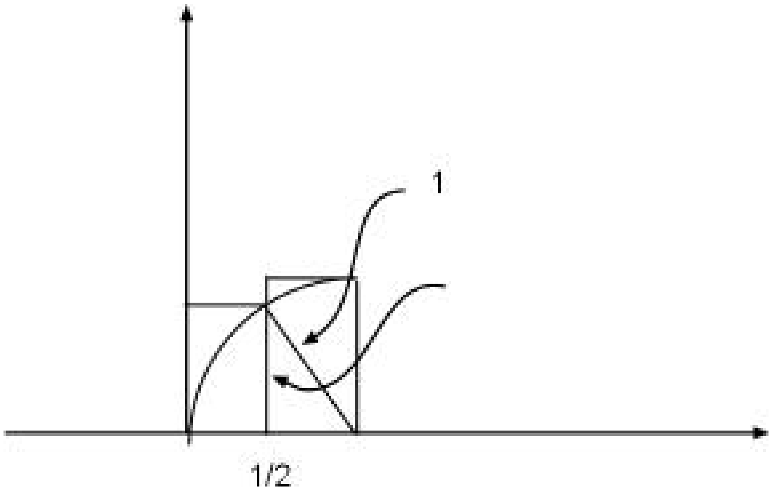

Figure 2:

In the n'th step the combined area of the 2^n *rectangles is*

$$\frac{1}{2^n}(1+\sqrt{1-(\frac{1}{2^n})^2}+\sqrt{1-(\frac{2}{2^n})^2}+\cdots+\sqrt{1-(\frac{2^n-1}{2^n})^2})$$

Thus we get the estimate: area of disc $\leq a_n = 4(\frac{1}{2^n}\sum_{i=0}^{i=2^n-1}\sqrt{1-(\frac{i}{2^n})^2})$. *We can see from the figures that the rectangles fit better and better around the disc and so w can continue this process and approximate the area of the disc as closely as we want. Using MATLAB we can easily compute the terms in this sequence (at least as many as our computing power allows)*

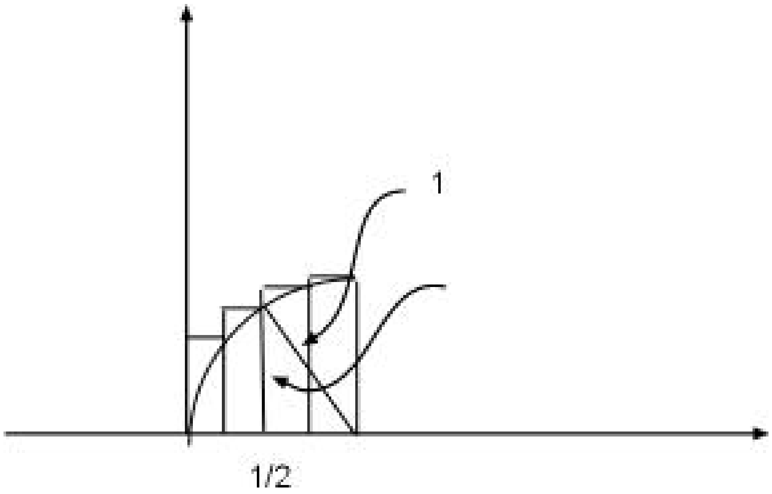

Figure 3:

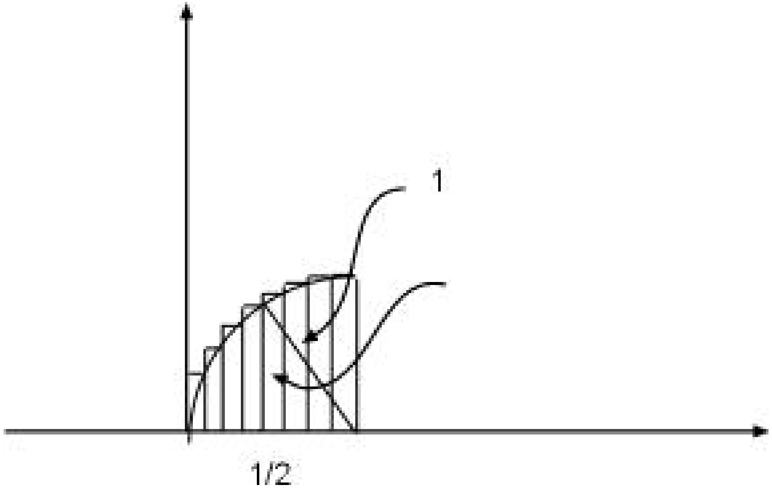

Figure 4:

Here are the 25 *first terms*

$n = 1$ $a_1 = 3.73205080756888$
$n = 2$ $a_2 = 3.49570906810244$
$n = 3$ $a_3 = 3.33981914435717$
$n = 4$ $a_4 = 3.24825303782774$
$n = 5$ $a_5 = 3.19760242287713$
$n = 6$ $a_6 = 3.17054691277968$
$n = 7$ $a_7 = 3.15640579239662$
$n = 8$ $a_8 = 3.14911808295723$
$n = 9$ $a_9 = 3.14539740271965$
$n = 10$ $a_{10} = 3.14350989153902$
$n = 11$ $a_{11} = 3.14255652791144$
$n = 12$ $a_{12} = 3.14207644885776$
$n = 13$ $a_{13} = 3.14183520817470$
$n = 14$ $a_{14} = 3.14171416315139$
$n = 15$ $a_{15} = 3.14165349049049$
$n = 16$ $a_{16} = 3.14162310107395$
$n = 17$ $a_{17} = 3.14160788759690$
$n = 18$ $a_{18} = 3.14160027422267$
$n = 19$ $a_{19} = 3.14159646518915$
$n = 20$ $a_{20} = 3.14159455984319$
$n = 21$ $a_{21} = 3.14159360687662$
$n = 22$ $a_{22} = 3.14159313029015$
$n = 23$ $a_{23} = 3.14159289195996$
$n = 24$ $a_{24} = 3.14159277278239$
$n = 25$ $a_{25} = 3.14159271318806$

Type in the following code as an M-file in MATLAB and use it to compute some of the numbers in this sequence:

```
function y=A(n)
v=0:2^ n;
S=sqrt(1-((2^ n-v)/2^ n).^ 2);
y=4*(1/2^ n)*sum(S);
```

MATLAB gives the following value for $\pi = 3.14159265358979$ *and as we*

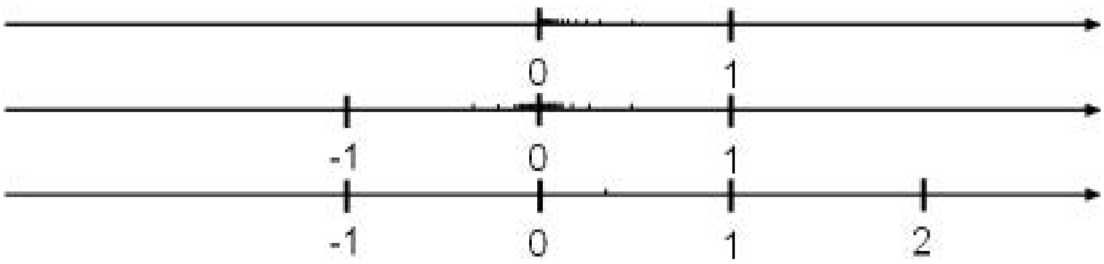

Figure 5:

know the area of the unit disc is π it certainly looks likely that the terms in this sequence approaches π as n becomes larger

The notion that a sequence of real numbers approach another real number is fundamental in calculus and we need a formal definition of what it means. Before we give the formal definition let us consider a few more examples of sequences.

Example 1.2 *1.* $a_n = \frac{1}{n}$

2. $b_n = (-1)^n \frac{1}{n}$

3. $c_n = \begin{cases} n \text{ if } n \text{ is even} \\ \frac{1}{n} \text{ if } n \text{ is odd} \end{cases}$

Thus the sequence $\{a_n\}$ *is* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots,$ *the sequence* $\{b_n\}$ *is* $-1, \frac{1}{2}.-\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^n \frac{1}{n}, \ldots$ *and* $\{c_n\}$ *is the sequence* $1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \frac{1}{9}, 10, \ldots$

We see that the terms in both $\{a_n\}$ *and* $\{b_n\}$ *"cluster" around* 0 *but while half of the terms of* $\{c_n\}$ *cluster around* 0 *the other half grows larger and larger.*

It is this clustering behavior we shall use to give a strict mathematical definition of what it means for a sequence of real numbers to "approach" a fixed real number a.

Definition 1.0.1 *Let* $\{a_n\}$ *be a sequence of real numbers. We say that* $\{a_n\}$ converges *to a real number* a *(notation:* $a_n \to a$ *for* $n \to \infty$ *or* $\lim_{n\to\infty} a_n = a$*), if no matter how close we are to* a*, all the numbers in the sequence except for at most finitely many are even closer to* a*. This is the definition in words. In mathematics we have to make precise the notion of being close.*

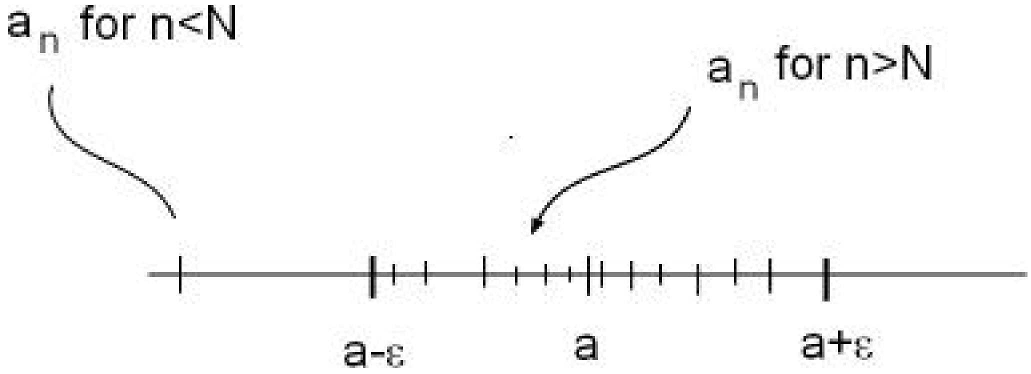

Figure 6:

Let ε be a positive real number (you should think of ε as a very small positive number). The numbers which are ε close to a consists of the real numbers x such that the distance between x and a, $|a-x|$ is smaller than ε. Thus the real numbers ε close to a is the set $\{x|a-\varepsilon < x < a+\varepsilon\} = (a-\varepsilon, a+\varepsilon)$. The definition of convergence means that no matter how small an $\varepsilon > 0$ we take all but finitely many terms in the sequence are ε close to a i.e. in $(a-\varepsilon, a+\varepsilon)$. In other words for each ε there will be a number N (depending on ε) such that when $n > N$, $|a-a_n| < \varepsilon$ i.e. from N onwards the terms of the sequence are ε close to a. Of course as we take smaller and smaller εs more and more of the sequence will lie outside $(a-\varepsilon, a+\varepsilon)$ but no matter how small an ε it will always be only a finite number.

Example 1.3 *In our area example, consider $\varepsilon = 0.0001$ then we see that from $n = 19$ onwards we have $|\pi - a_n| < \varepsilon$. Remark that this does not prove that $a_n \to \pi$. In order to do that we would have to show that for any ε we can find N (depending on ε of course) such that from $n = N$ onwards $|\pi - a_n| < \varepsilon$. We have only proved it for a single ε*

Let's formally prove that $\dfrac{1}{n} \to 0, n \to \infty$.

Consider any $\varepsilon > 0$ and consider $\dfrac{1}{\varepsilon}$ (if ε is very small this number will be very large). Choose a large natural number N such that $N > \dfrac{1}{\varepsilon}$. Then for any $n > N$ we have $n > \dfrac{1}{\varepsilon}$ and hence $\dfrac{1}{n} < \varepsilon$. This proves that from N onwards $|\dfrac{1}{n} - 0| = \dfrac{1}{n} < \varepsilon$, which is precisely what it means for $\dfrac{1}{n} \to 0$.

Homework Problems (due Monday 10/6):

1. Show that for any two real numbers a, b *the following inequality holds:* $|a+b| \leq |a| + |b|$ *(Hint: consider separately the four cases* $a > 0, b > 0; a < 0, b > 0; a < 0, b < 0; a > 0, b < 0$*)*

2. Show that for any two real numbers a, b*,* $||a| - |b|| \leq |a - b|$

3. Show that for any real number a *the sequence* $a + \dfrac{1}{n} \to a$

4. Show that if $\{a_n\}$ *and* $\{b_n\}$ *are sequences such that* $a_n \to a$ *and* $b_n \to b$ *then the sequence* $\{a_n + b_n\}$ *converges to* $a + b$*, and the sequence* $\{a_n b_n\}$ *converges to* ab