7.1 Power series of Complex Numbers

Consider a sequence of complex numbers $\{z_n\}_n$ and form a new sequence $\{s_n\}_n$ where $s_n = z_1 + z_2 + \cdots + z_n$, the sequence of partial sums. If the sequence $\{s_n\}_n$ is convergent then we say that the infinite series $\sum_{i=1}^{\infty} z_i$ is convergent. Thus it makes perfect sense to talk about infinite series of complex numbers and of convergence of such.

Definition 7.1.1 *Consider a series* $\sum_{i=1}^{\infty} z_i$*. If the series* $\sum_{i=1}^{\infty} |z_i|$ *(remark that this is a series of* real numbers*) is convergent, then the series* $\sum_{i=1}^{\infty} z_i$ *is said to be* absolutely convergent

The essential result is again that absolute convergence implies convergence:

Theorem 7.1.1 *Consider a series of complex numbers* $\sum_{i=1}^{\infty} z_i$*. Assume the series is absolutely convergent. Then it is convergent.*

Proof: Consider the sequence of partial sums $s_n = z_1 + z_2 + \cdots + z_n$. We have to show that this sequence is convergent. Recall from Lemma 7.0.3 that $\{s_n\}_n$ is convergent if and only if the two sequences of *real* numbers $a_n = \Re s_n$ and $b_n = \Im s_n$ are convergent. But to show that a sequence of real numbers is convergent we only have to verify that it is a Cauchy sequence.

Write $z_i = c_i + i \cdot d_i$ so $c_i = \Re z_i$ and $d_i = \Im z_i$. Then $s_n = z_1 + z_2 + \cdots + z_n = (c_1 + id_1) + (c_2 + id_2) + \cdots + (c_n + id_n) = a_n + ib_n$ and hence $a_n = c_1 + c_2 + \cdots + c_n$ and $b_n = d_1 + d_2 + \cdots + d_n$. We need to verify that the two sequences $\{a_n\}_n$ and $\{b_n\}_n$ are Cauchy sequences so we need to estimate the differences $|a_n - a_m|$ and $|b_n - b_m|$. Assume $n > m$ then $a_n - a_m = c_{m+1} + c_{m+2} + \cdots + c_n$ and $b_n - b_m = d_{m+1} + d_{m+2} + \cdots + d_n$. Let t_n denote the n'th partial sum of the absolute series $\sum_{i=1}^{\infty} |z_i|$. By assumption $\{t_n\}_n$ is convergent and hence it is a Cauchy sequence. We have $|t_n - t_m| = |z_{m+1}| + |z_{m+2}| + \cdots + |z_n| = \sqrt{c_{m+1}^2 + d_{m+1}^2} + \sqrt{c_{m+2}^2 + d_{m+2}^2} + \cdots + \sqrt{c_n^2 + d_n^2}$. Clearly $\sqrt{c_i^2 + d_i^2} \geq$ both $|c_i|$ and $|d_i|$ and so we have $\sqrt{c_{m+1}^2 + d_{m+1}^2} + \sqrt{c_{m+2}^2 + d_{m+2}^2} + \cdots + \sqrt{c_n^2 + d_n^2} \geq |c_{m+1}| + |c_{m+2}| + \cdots + |c_n| \geq |c_{m+1} + c_{m+2} + \cdots + c_n| = |a_n - a_m|$ and $\sqrt{c_{m+1}^2 + d_{m+1}^2} + \sqrt{c_{m+2}^2 + d_{m+2}^2} + \cdots + \sqrt{c_n^2 + d_n^2} \geq |d_{m+1}| + |d_{m+2}| + \cdots + |d_n| \geq |d_{m+1} + d_{m+2} + \cdots + d_n| = |b_n - b_m|$.

Thus we have shown that $|t_n - t_m| \geq \begin{cases} |a_n - a_m| \\ |b_n - b_m| \end{cases}$. Now let $\varepsilon > 0$ be given. Because $\{t_n\}$ is a Cauchy sequence we can find N such that when $n, m > N$ we have $|t_n - t_m| < \varepsilon$. But then the inequalities above show that we have both $|a_n - a_m| < \varepsilon$ and $|b_n - b_m| < \varepsilon$. Thus $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences and hence they are convergent, and the theorem is proved.

Remark that the notion of a Cauchy sequence makes perfect sense for complex numbers: a Cauchy sequence $\{w_i\}$ of complex numbers is a sequence with the property that for all $\varepsilon > 0$ there exists N such that when $n, m > N$, the distance $|w_n - w_m| < \varepsilon$. In the process of proving the theorem above we actually also proved that a Cauchy sequence of complex numbers is convergent i.e. the complex numbers are complete.

Consider now a complex power series i.e. a series of the form $\sum_{j=0}^{\infty} a_j z^j$. Taking absolute values and using that $|z^j| = |z|^j$, we get a real power series $\sum_{j=1}^{\infty} |a_j||z|^j$. To determine convergence properties of this power series we can use all our results from the real case. In particular we get a radius of convergence $\rho = \lim_{i\to\infty} \frac{|a_i|}{|a_{i+1}|}$. Thus the power series is absolutely convergent for $|z| < \rho$, i.e. for z in the open disc centered at 0 with radius ρ.

Example 7.6 *We have seen that the power series* $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ *is absolutely convergent for every* $x \in \mathbb{R}$ *and its limit is* $\exp(x)$ *i.e.* $\sum_{i=0}^{\infty} \frac{x^i}{i!} = \exp(x)$ *for all* $x \in \mathbb{R}$.

Now consider the complex power series $\sum_{i=0}^{\infty} \frac{z^i}{i!}$ *where* $z \in \mathbb{C}$. *Taking absolute values we get the series* $\sum_{i=0}^{\infty} \frac{|z^i|}{i!}$ *and since* $|z^i| = |z|^i$, *the absolute series is just the real power series* $\sum_{i=0}^{\infty} \frac{|z|^i}{i!}$, *which we know to be convergent. Thus we conclude that the complex power series* $\sum_{i=0}^{\infty} \frac{z^i}{i!}$ *is absolutely convergent, and hence convergent, for every* $z \in \mathbb{C}$. *We can then use the power series to define a function on the complex numbers* $\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!}$. *Remark that if* z *happens to be real then* $\exp(z)$ *is the usual exponential function.*

Example 7.7 *How can we extend the trigonometric functions* cos *and* sin *to the complex numbers?*

The power series are given by

$$\cos(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^i \frac{x^{2i}}{(2i)!)} + \dots$$

and

$$\sin(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^i \frac{x^{2i+1}}{(2i+1)!} + \dots$$

We simply define $\cos(z)$ *where* $z \in \mathbb{C}$ *by the power series so*

$$\cos(z) = \sum_{i=0}^{\infty} (-1)^i \frac{z^{2i}}{(2i)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^i \frac{z^{2i}}{(2i)!)} + \dots$$

Similarly

$$\sin(z) = \sum_{i=0}^{\infty} (-1)^i \frac{z^{2i+1}}{(2i+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^i \frac{z^{2i+1}}{(2i+1)!} + \dots$$

Since the real power series for cos *and* sin *are absolutely convergent everywhere, the complex power series are also absolutely convergent, and hence convergent everywhere. Thus we have extended* cos *and* sin *to the complex numbers*

Example 7.8 *Consider a complex number of the form* $z = it$ *where* t *is a real number, i.e.* $\Re z = 0$*. We want to compute* $\exp(it)$*. Thus we plug it into the power series*

$$\exp(it) = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots + \frac{(it)^n}{n!} + \dots$$

Now $(it)^n = i^n t^n = \begin{cases} (-1)^j t^n \text{ for } n = 2j \\ i(-1)^j t^n \text{ for } n = 2j+1 \end{cases}$*. Hence*

$$\exp(it) = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \frac{t^6}{6!} + \cdots + (-1)^j \frac{t^{2j}}{(2j)!} + i(-1)^j \frac{t^{2j+1}}{(2j+1)!} + \dots$$

Since the series is absolutely convergent we can rearrange the terms any way we see fit without changing the limit and so by summing the even and the odd terms separately we get

$$\exp(it) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$$

We recognize the power series $1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$ *as* $\cos(t)$ *and* $t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$ *as* $\sin(t)$*. Thus we get the following amazing formula:*

$$\exp(it) = \cos(t) + i\sin(t)$$

This known as Euler's formula

We shall further investigate the complex exponential function. The real exponential function satisfies the identity $\exp(x_1+x_2) = \exp(x_1)\exp(x_2)$ for all $x_1, x_2 \in \mathbb{R}$. The analogous property for the complex exponential function would be $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

To prove this we first prove a lemma which is useful for other purposes as well:

Lemma 7.1.1 *Consider a power series* $g(z) = \sum_{i=0}^{\infty} a_i z^i$*. Let the power series be absolutely convergent in the open disc* $\{z \in \mathbb{C} \| z| < \rho\}$*. Assume that* $g(x) = \sum_{i=0}^{\infty} a_i x^i = 0$ *for all* $x \in \mathbb{R}$ *with* $|x| < \rho$*. Then* $g(z) = 0$ *in the whole disc*

Proof: Assume first that all the coefficients (the a_i's) are real numbers. Consider the open interval $I = \{x| - \rho < x < \rho\}$. The power series defines a function $g : I \to \mathbb{R}$ (here we use that the coefficients are real numbers, otherwise we would not have $g(x) \in \mathbb{R}$ for $x \in \mathbb{R}$) and we know that this function is infinitely often differentiable and $g^(i)(0) = i!a_i$. Since g is identically 0 in the open interval, all the derivatives of g at 0 must also vanish and so $a_i = 0$ for all i. But then $g(z) = 0 + 0 \cdot z + 0 \cdot z^2 + 0 \cdot z^3 + \dots = 0$ for every z. This proves the lemma when the coefficients are real numbers.

In the general case, write $\alpha_i = \Re a_i$ and $\beta_i = \Im a_i$. Then we have $a_i = \alpha_i + i\beta_i$ and $g(x) = \sum_{i=0}^{\infty} (\alpha_i + i\beta_i) x^i$. We have $|\alpha_i||x|^i \le |a_i||x|^i$ and $|\beta_i||x|^i \le |a_i||x|^i$ and since $g(x)$ is absolutely convergent we can use the comparison test to conclude that both $g_1(x) = \sum_{i=0}^{\infty} \alpha_i x^i$ and $g_2(x) = \sum_{i=0}^{\infty} \beta_i x^i$ are absolutely convergent. Since the series is absolutely convergent we can rearrange the terms so we can write $g(x) = g_1(x) + ig_2(x)$. Thus $g_1(x) = \Re g(x) = 0$ and $g_2(x) = \Im g(x) = 0$. Now we can apply the argument above to g_1 and g_2 to conclude that $\alpha_i = \beta_i = 0$ for all i and hence we get $g(z) = 0$ for every z in the open disc.

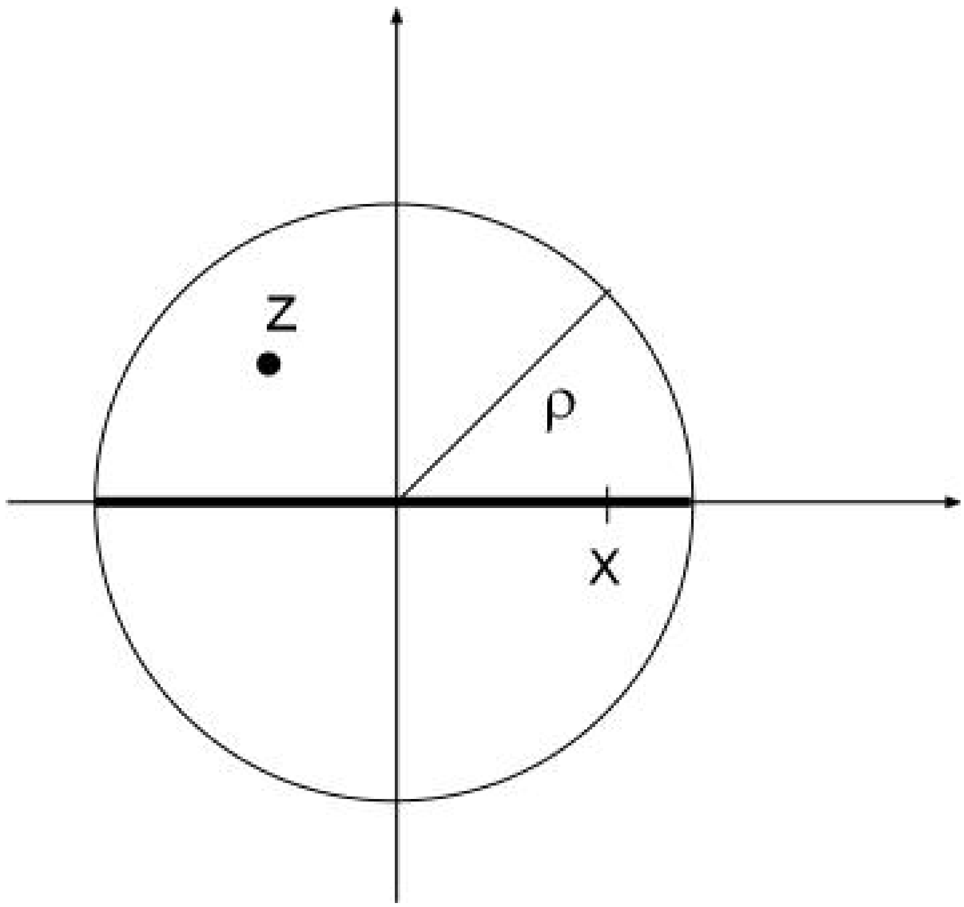

Figure 82:

To prove the formula, fix $z_2 = x_2$ a real number. Consider the function $\exp(z + x_2) - \exp(z)\exp(x_2)$. This is defined by the power series $\sum_{i=0}^{\infty} \frac{(z + x_2)^i}{i!} - \sum_{i=0}^{\infty} \exp(x_2)\frac{x_2^i}{i!}$. We know that $\exp(x+x_2)-\exp(x)\exp(x_2) = 0$ for all real x. Hence by the lemma we can conclude that $\exp(z + x_2) - \exp(z)\exp(x_2) = 0$. Thus the formula is true if either z_1 or z_2 is real.

Now fix z_1 and consider the function $\exp(z_1 + z) - \exp(z_1)\exp(z)$ as a function of z. Since the formula holds when $z \in \mathbb{R}$, this function is identically 0 on the real numbers. Again by the lemma we conclude it is identically 0 on the complex numbers.

Example 7.9 *Find formulas for* $\cos(\theta_1 + \theta_2)$ *and* $\sin(\theta_1 + \theta_2)$.

We did this using geometry back when we computed the derivatives of cos *and* sin *but it was not very elegant. Now we can do this easily using Euler's formula:* $\exp(i(\theta_1+\theta_2)) = \cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$. *By the formula above* $\exp(i\theta_1+i\theta_2) = \exp(i\theta_1)\exp(\imath\theta_2) = (\cos(\theta_1)+i\sin(\theta_1))(\cos(\theta_2)+i\sin(\theta_2)) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1))$. *Then we can just take the real and imaginary parts and we get* $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)$ *and* $\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)$

We can also use Euler's formula to find another representation of a complex number. So far we have two ways of representing complex numbers: as a vector $(a, b) \in \mathbb{R}^2$ or as an expression $a + ib$.

Now a point in $(a, b) \in \mathbb{R}^2$ is uniquely determined by two quantities: the length of the vector, $r = \sqrt{a^2 + b^2}$ and the angle, θ this vector forms with the x-axis. Here $0 \le \theta < 2\pi$. The pair (r, θ) is called the polar coordinates of the complex number.

If $z = a + ib$, then clearly $r = |z|$, the angle θ can be determined by $\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}$. Consider now $\frac{z}{|z|}$. This complex number has absolute value $= 1$, hence is a point on the unit circle. Thus $\frac{z}{|z|}$ is of the form $(\cos(\theta), \sin(\theta)) = \cos(\theta) + i\sin(\theta) = \exp(i\theta)$. Thus we have $z = |z|\exp(i\theta)$. The number $|z|$ is sometimes called the *modulus* of z and the angle θ the *argument* of z, Arg z. Hence we can write $z = |z|\exp(i\text{Arg z})$.

The polar coordinates are convenient for multiplying complex numbers: if $z_1 = |z_1|\exp(i\theta_1)$ and $z_2 = |z_2|\exp(i\theta_2)$ then $z_1z_2 = |z_1||z_2|\exp(i\theta_1)\exp(i\theta_2) = |z_1||z_2|\exp(i(\theta_1 + \theta_2))$. Thus the modulus of a product is the product of the moduli and the argument of the product is the sum of the arguments.

Consider a complex number $z = a + ib$. Then $\exp z = \exp(a + ib) = \exp(a)\exp(ib)$. Thus $|\exp z| = \exp(\Re z)$ and the argument of $\exp z$ is b.

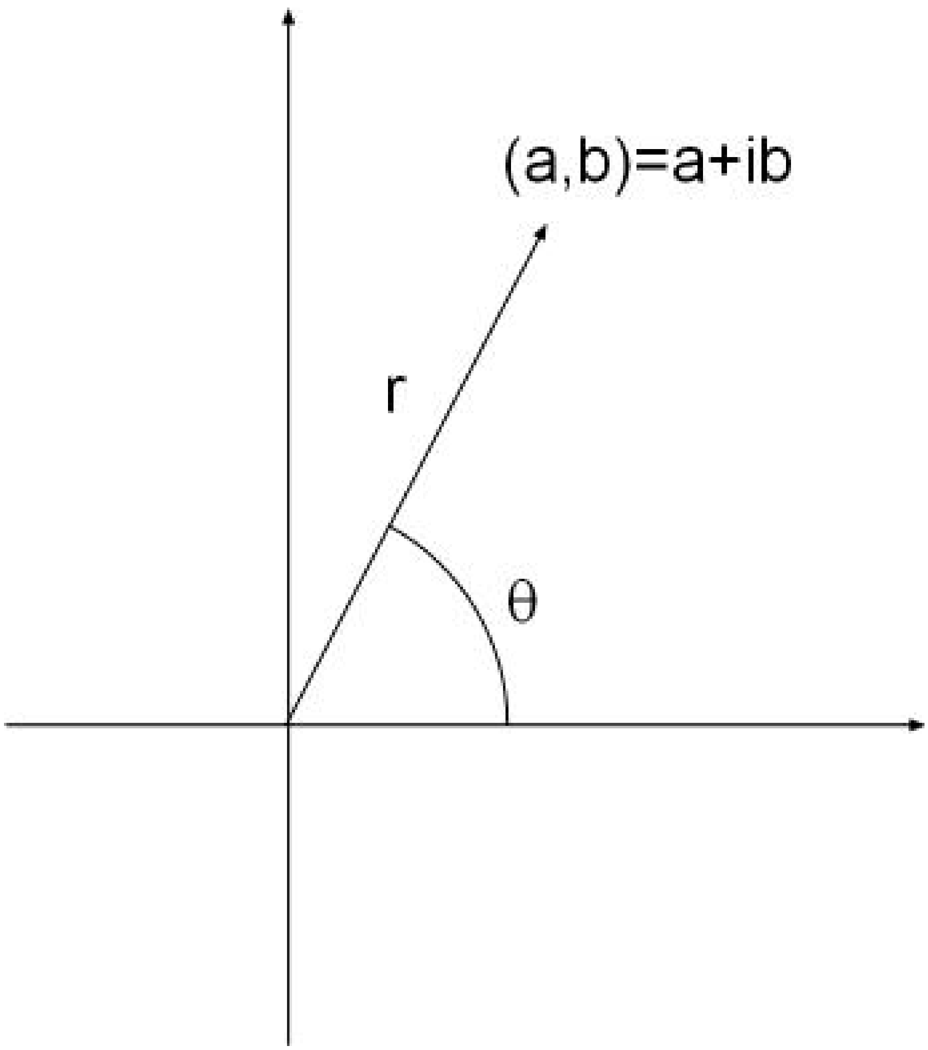

Figure 83: Polar coordinates

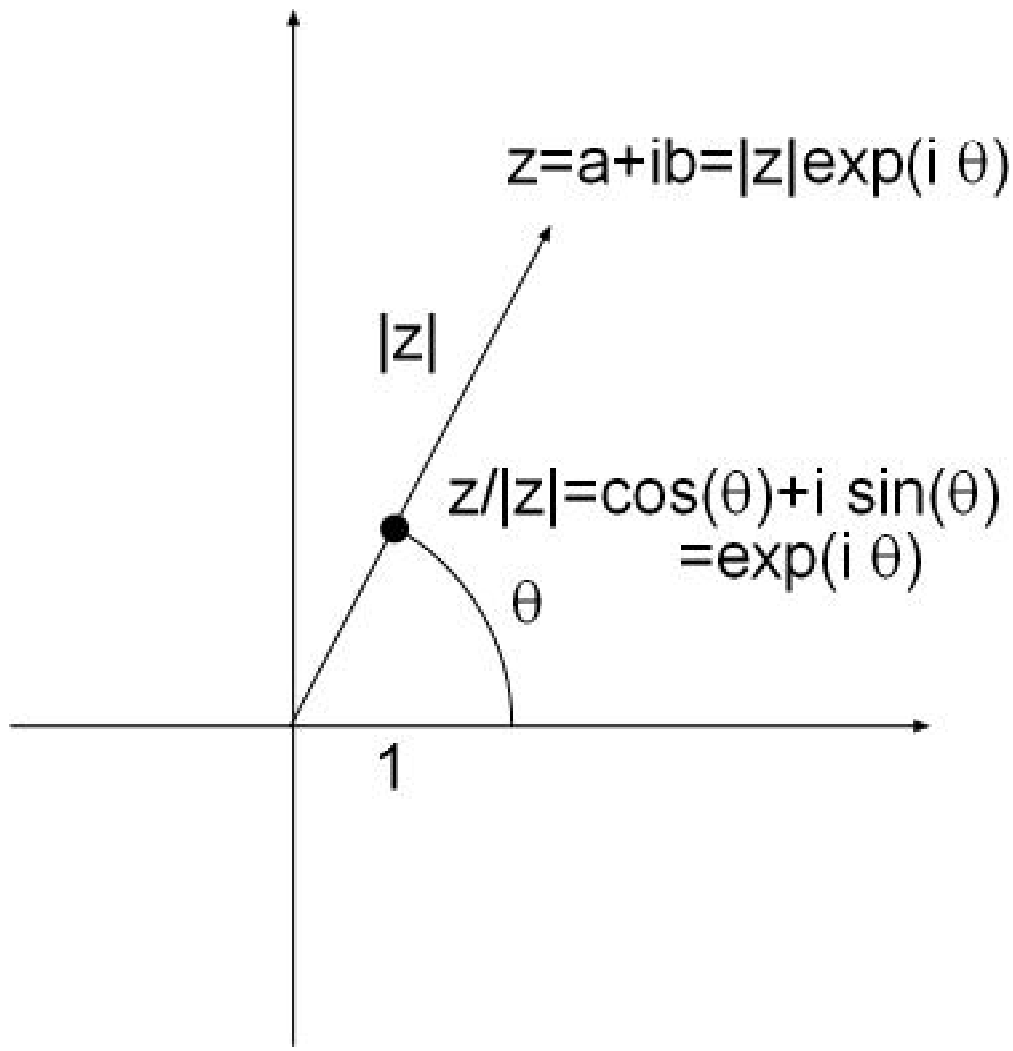

Figure 84:

To define a complex logarithm we recall that in the real numbers $\log(\exp x) = x$ and for $x > 0$, $\exp(\log x) = x$. Thus for a complex number it is natural to define $\log z$ such that $\exp(\log z) = z$. If $\log z = u + iv$ we have $\exp(u + iv) = \exp(u)\exp(iv) = a + ib$. Hence $\exp(u) = |z|$ so $u = \log(|z|)$ and $v = \text{Arg } z$. Clearly z cannot be 0. But we can put $\log z = \log |z| + i(\text{Arg } z + 2n\pi$ for any n so the equation $\exp(\log z) = z$ does not define a single value for $\log z$ but rather infinitely many. This is because the complex exp function is not $1 - 1$, indeed $\exp(a + ib) = \exp(a + i(b + 2n\pi))$ for all n. For instance $1 = \exp(i2\pi) = \exp(i4\pi) = \exp(2n\pi)$ for any integer n.

Thus it is not possible to define a single value log function everywhere: we have to make a choice of where we take the argument of $\log z$. We can take any half-open interval of length 2π. Suppose we take the argument to lie in the interval $[0, 2\pi]$. Consider the point 1 and consider the two sequences $z_n = 1 + i\frac{1}{n}$ and $w_n = 1 - i\frac{1}{n}$, both converging to 1. Then $\text{Arg } z_n \to 0$ and $\text{Arg } w_n \to 2\pi$. Hence $\log(z_n) \to 0$ and $\log(w_n) \to i2\pi$. This shows that log with this definition is not continuous at 1, in fact it is not continuous anywhere on the non-negative real line. Thus to avoid discontinuities we can only define log on the complex plane -the non-negative real numbers.

If we take the argument to lie in the open interval $]-\pi, \pi[$ we can define a log function on $\mathbb{C} - \{x \in \mathbb{R} | x \leq 0\}$. Then when z is a positive real number $\log z$ agrees with the usual real logarithm function. Remark that with this definition $-\pi < \Im \log z < \pi$ so log maps the whole complex plane into a strip of with 2π centered around the x-axis.

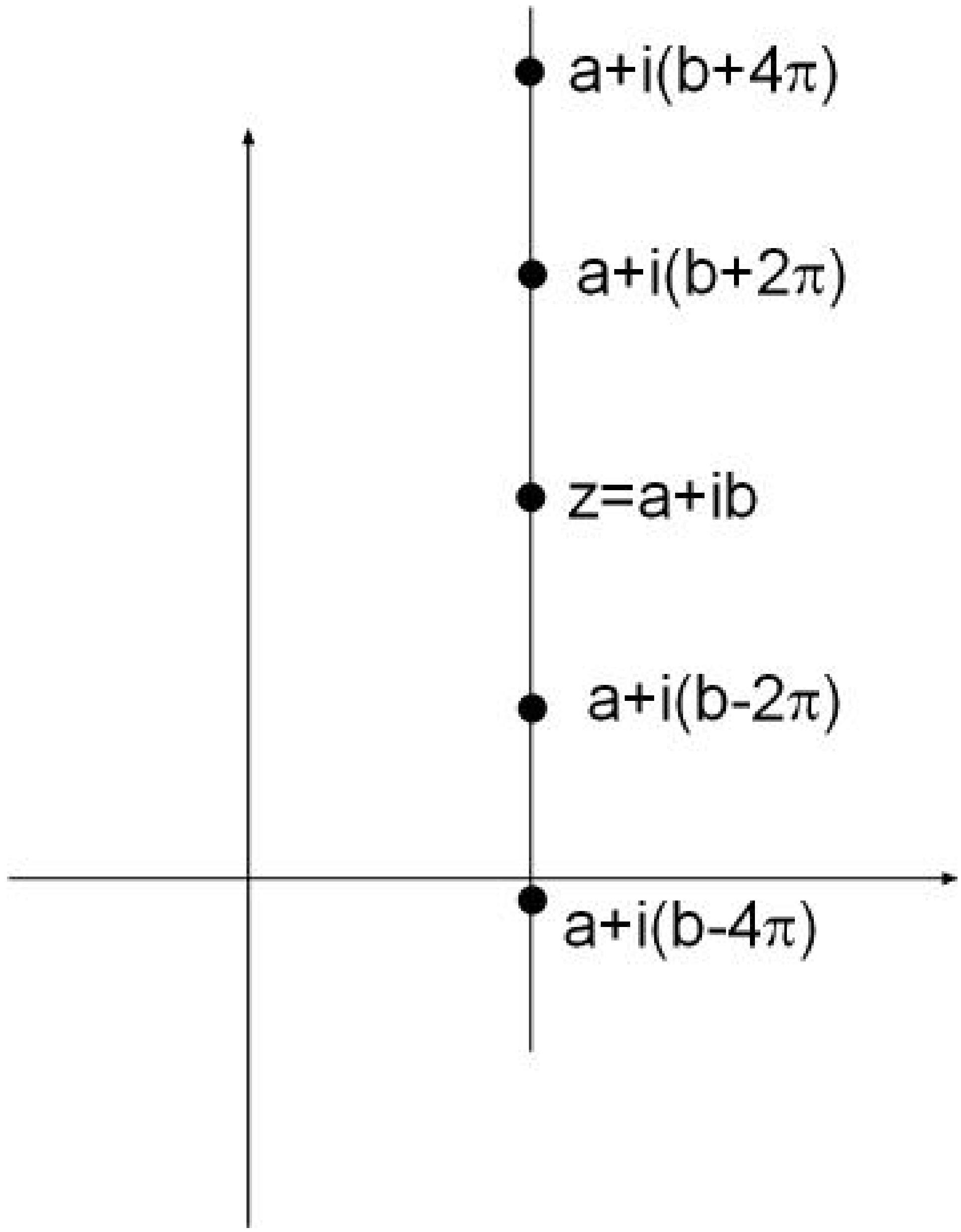

Figure 85:

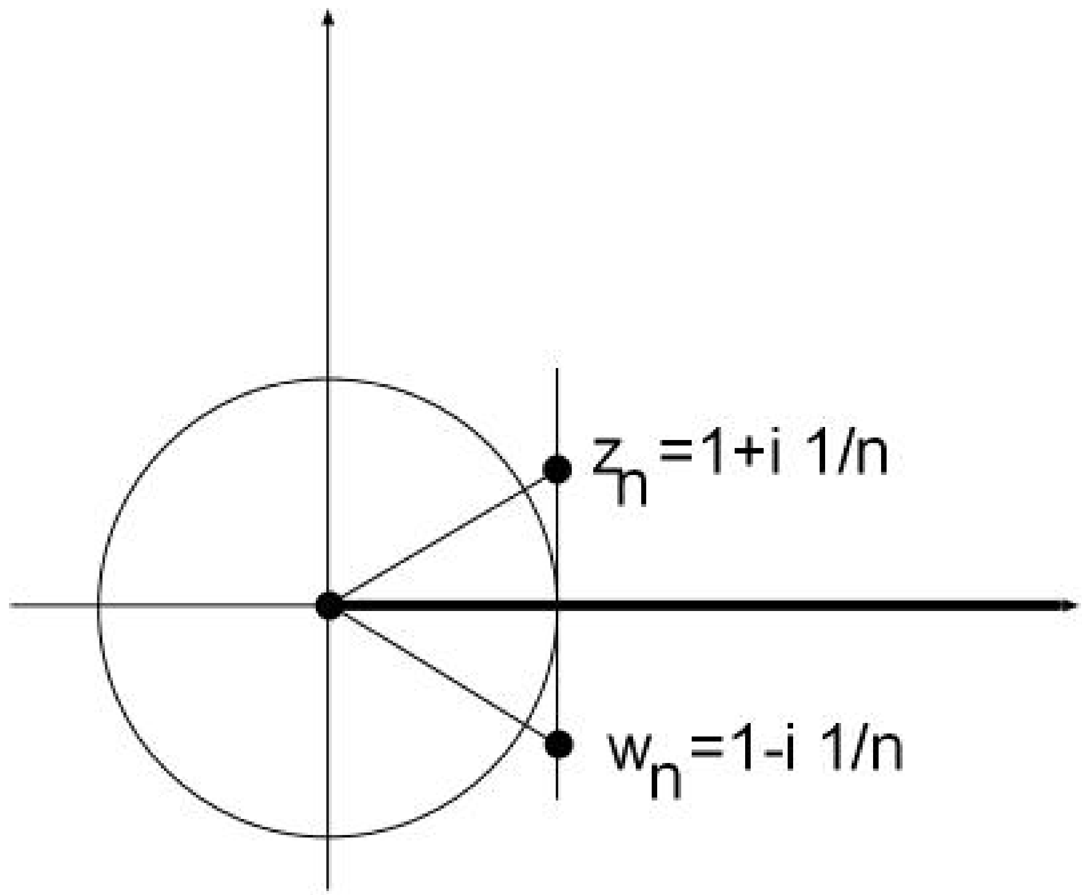

Figure 86:

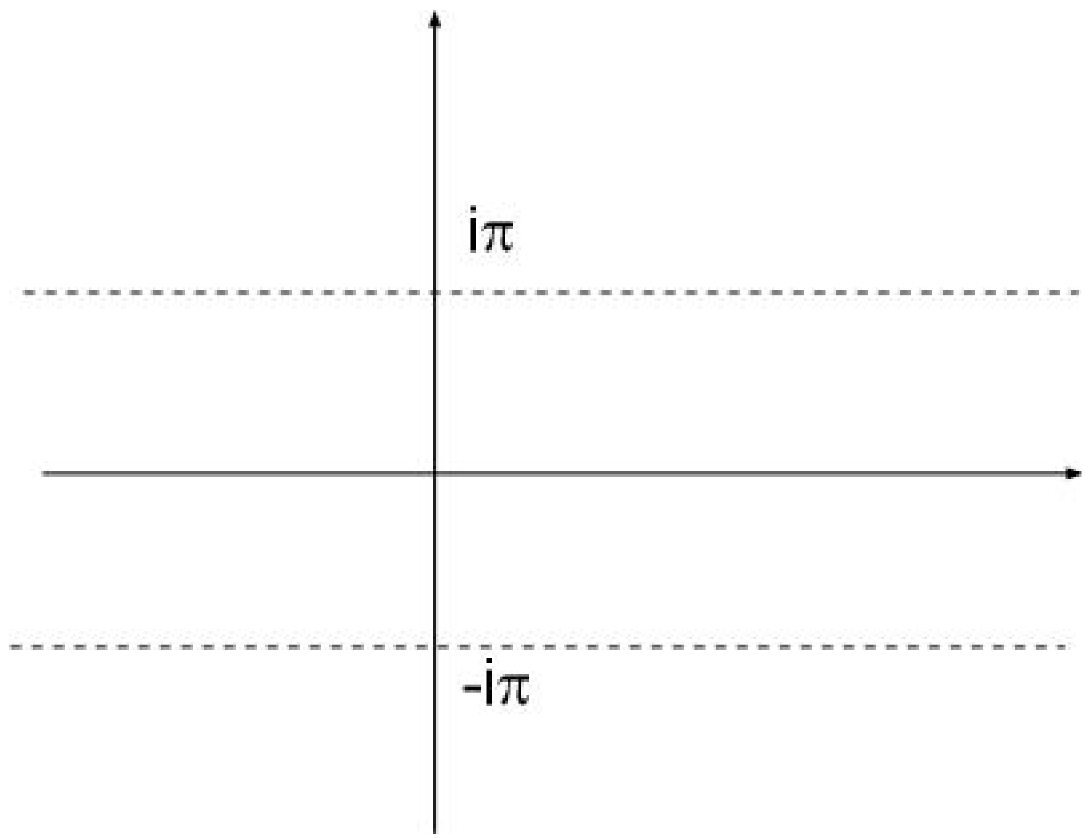

Figure 87:

Homework Problems due 2/9-2004

Problem 1. Use the formula $\exp(i\theta)^n = \exp(in\theta)$ to find formulas for $\cos(2\theta), \cos(3\theta)$ and $\cos(4\theta)$

Problem 2. Use the formula for the sum of a geometric progression to compute

$$\exp(i\theta) + \exp(i2\theta) + \cdots + \exp(in\theta)$$

and find formulas for the trigonometric sums

$$cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta)$$

and

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta)$$

Problem 3. Write the following complex numbers in the form $r\exp(i\theta)$

- $1 + i$
- $2i$
- -1
- $1 - i$
- $-1 + i$
- $-1 - i$