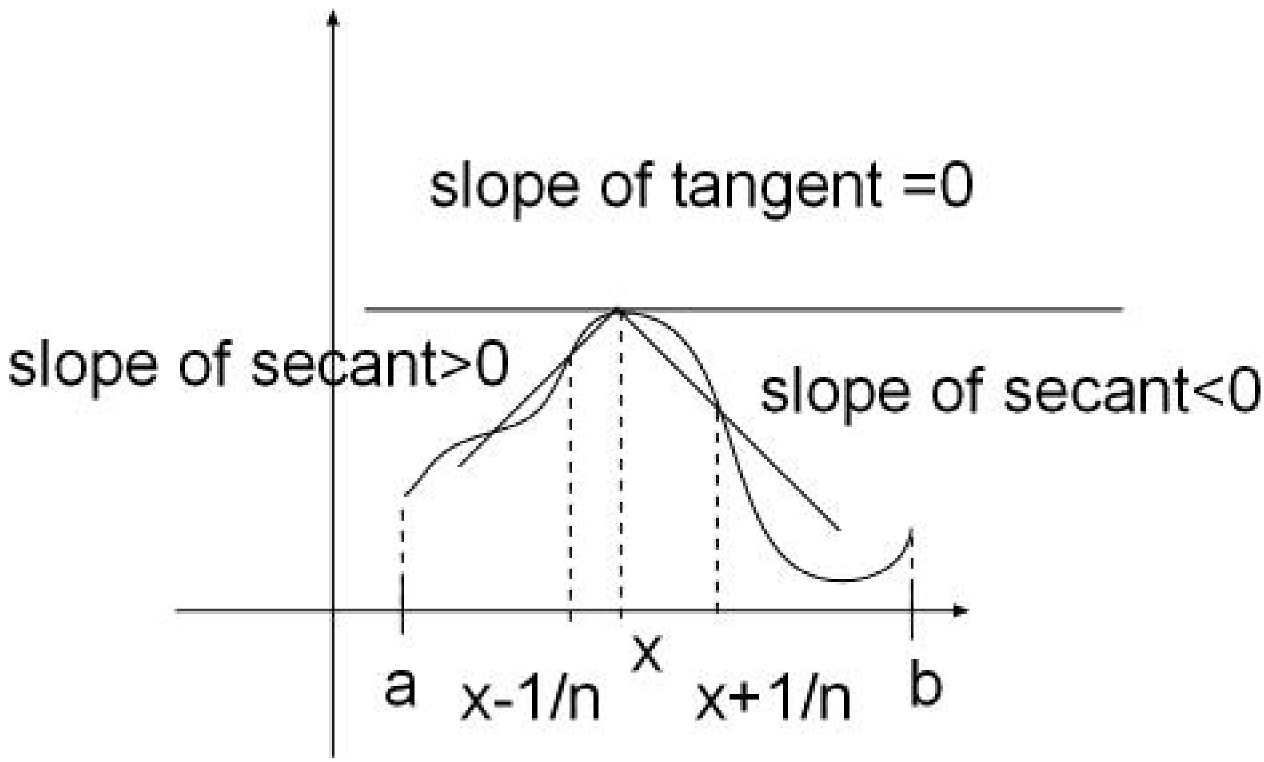

Figure 38:

3.1 The Mean Value Theorem for Differentiable Functions

Consider a function $f : [a, b] \to \mathbb{R}$. We assume that f is differentiable at every point of the open interval $]a, b[$ and is continuous on the closed interval $[a, b]$.

We know that f takes a maximum value and a minimum value. These points may be at the end points a and b, or they may be in the open interval $]a, b[$.

Lemma 3.1.1 *Assume $x \in]a, b[$ is a maximum (resp. minimum) point (i.e. the function attains its maximum (resp. minimum) at x. Then $f'(x) = 0$.*

Proof: Consider the sequence $x + \dfrac{1}{n}$. This sequence converges to x

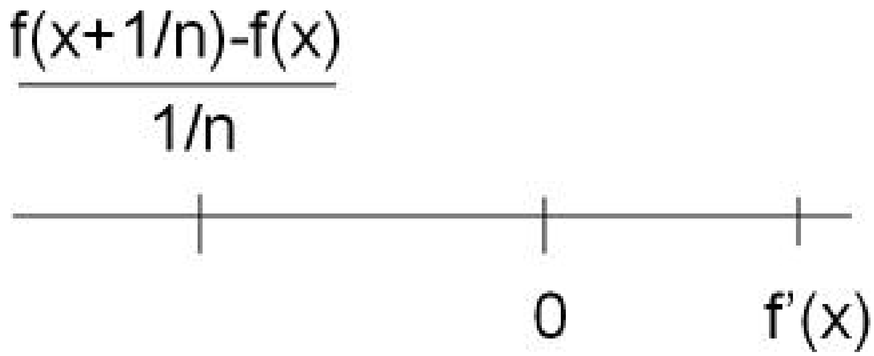

Figure 39:

and since f is differentiable $\dfrac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} \to f'(x)$. Since $f(x)$ is the maximum value it is larger than all the other values of f and so $f(x+\frac{1}{n}) \leq f(x)$. Thus $f(x+\frac{1}{n})-f(x) \leq 0$ and hence $\dfrac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} \leq 0$ for all n and so the limit $f'(x)$ is also ≤ 0. Now consider the sequence $x-\frac{1}{n}$. This sequence also converges to x and $f(x-\frac{1}{n})-f(x) \leq 0$. Now the difference in the $x-$values is $-\frac{1}{n}$ and so $\dfrac{f(x-\frac{1}{n})-f(x)}{-\frac{1}{n}} \to f'(x)$, but $\dfrac{f(x-\frac{1}{n})-f(x)}{-\frac{1}{n}} \geq 0$ so $f'(x) \geq 0$. But this means that $f'(x) \leq 0$ and $f'(x) \geq 0$ and the only way this can be true is if $f'(x) = 0$. The same argument works if x is a minimum point.

We shall now apply this result to show that a function f as above which takes the value 0 at both endpoints i.e. $f(a) = 0 = f(b)$ must have a horizontal tangent at some point in the open interval $]a,b[$ i.e. there must be a point x in $]a,b[$ such that $f'(x) = 0$. This theorem is known as Rolle's Theorem.

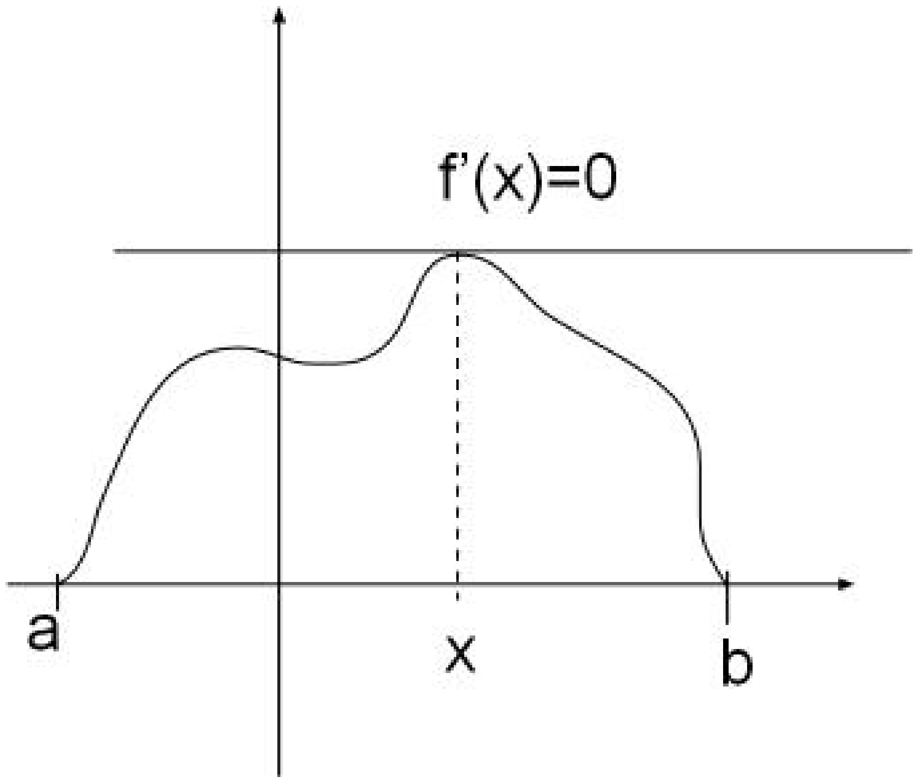

Figure 40:

Theorem 3.1.1 *(Rolle's Theorem) Let f be differentiable on the open interval $]a, b[$ and continuous on the closed interval $[a, b]$. Assume $f(a) = 0 = f(b)$. Then there exits a point $x \in]a, b[$ such that $f'(x) = 0$*

Proof: This is almost a triviality. If f is constant equal to 0 on the whole interval then it is clear. So assume there is a point z such that $f(z) \neq 0$. Assume $f(z) > 0$. Since f is continuous on the closed interval $[a, b]$ it must attain a maximum, say at x. Since $f(z) > 0$ the maximum value of f must also be positive i.e. $f(x) > 0$ because $f(x) \geq f(z) > 0$. Then x cannot be one of the endpoints because there the value of the function is 0 thus $x \in]a, b[$. Since x is a maximum point we get $f'(x) = 0$. The same argument works if $f(z) < 0$. In this case we take x to be a minimum point, then $f(x) \leq f(z) < 0$ and again x cannot be an endpoint, also since x is a minimum point $f'(x) = 0$

Now we are ready to prove the Mean Value Theorem for Differentiable Functions:

It basically states that at some point in the open interval the tangent is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$. If $f(a) = 0 = f(b)$ then the secant line is just the $x-$axis and hence the MVT in this case states that there is a point in $]a, b[$ where the tangent to the graph is horizontal, in other words Rolle's Theorem. What is more interesting is that as we shall see the MVT is a consequence of Rolle's Theorem.

Theorem 3.1.2 *Let f be continuous on $[a, b]$ and differentiable on $]a, b[$. Consider the secant line through the points $(a, f(a))$ and $(b, f(b))$. The slope of this line is $\frac{f(b) - f(a)}{b - a}$. There exists a point $x_0 \in]a, b[$ such that* $\frac{f(b) - f(a)}{b - a} = f'(x_0)$

Proof: The proof is amazingly simple: consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - f(a) - (x - a)\frac{f(b) - f(a)}{b - a}$. This function is continuous on $[a, b]$ and differentiable on $]a, b[$ because it is a sum of functions with those properties. Now $g(a) = f(a) - f(a) - (a - a)\frac{f(b) - f(a)}{b - a} = 0$ and $g(b) = f(b) - f(a) - (b - a)\frac{f(b) - f(a)}{b - a} = f(b) - f(a) - (f(b) - f(a)) = 0$. Thus g is 0 at the endpoints and so we can apply Rolle's Theorem, which says that there exists a point $x_0 \in]a, b[$ such that $g'(x_0) = 0$.

Now we compute g'. Using the sum rule $g'(x) = f'(x) - (x - a)'\frac{f(b) - f(a)}{b - a}$, but the derivative of the function $x - a$ is 1 and so $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$. Now $g'(x_0) = 0$ is equivalent to $f'(x_0) = \frac{f(b) - f(a)}{b - a}$

Another variant of the MVT is what is known as the Cauchy MVT (same Cauchy as in Cauchy sequence). It states:

Theorem 3.1.3 *Assume f and g are continuous on $[a, b]$ and differentiable on $]a, b[$. Assume $g(a) \neq g(b)$ Then there exists $x_0 \in]a, b[$ such that* $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$

Proof: The trick is the same as before: find a function that we can apply Rolle's theorem to. By some trial and error we get to the function h defined

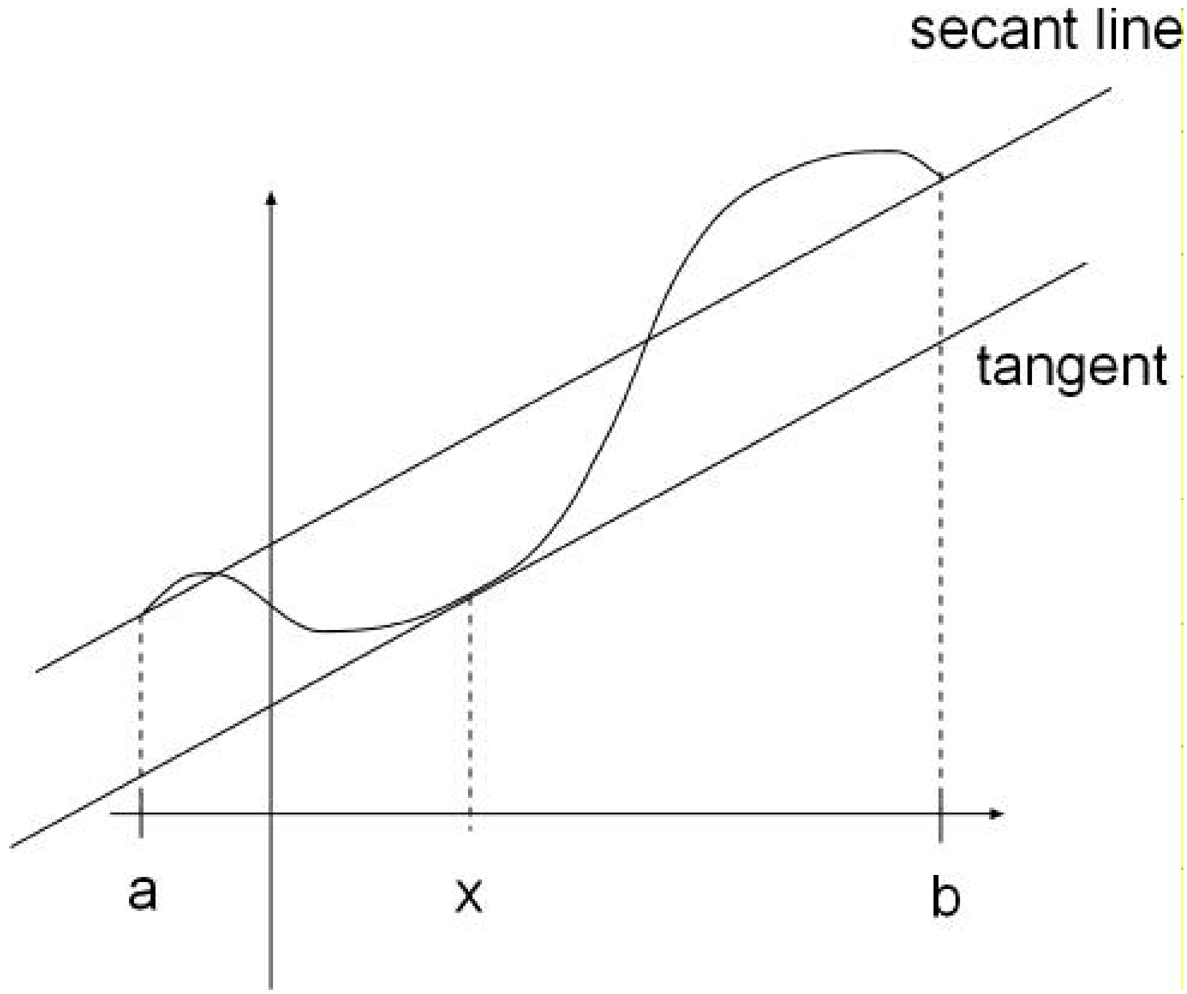

Figure 41:

by $h(x) = (g(x) - g(a))(f(b) - f(a)) - (f(x) - f(b))(g(b) - g(a))$. Then h is continuous on $[a, b]$ and differentiable on $]a, b[$. Also $h(a) = (g(a) - g(a))(f(b) - f(a)) - (f(a) - f(a))(g(b) - g(a)) = 0$ and $h(b) = (g(b) - g(a))(f(b) - f(a)) - (f(b) - f(a))(g(b) - g(a)) = 0$. Thus Rolle's Theorem applies and there exists an $x_0 \in]a, b[$ such that $h'(x_0) = 0$. Now $h'(x) = (g(x)-g(a))'(f(b)-f(a))-(f(x)-f(a))'(g(b)-g(a))$. But $(g(x)-g(a))' = g'(x)$ and $(f(x) - f(b))' = f'(x)$ so $h'(x) = g'(x)(f(b) - f(a)) - f'(x)(g(b) - g(a))$ and so at x_0 we have $g'(x_0)(f(b) - f(a)) = f'(x_0)(g(b) - g(a))$ which is precisely what we wanted to prove.

Here is a nice application of the Cauchy MVT. Let f and g be differentiable functions and assume the derivatives are continuous. Let a be a point such that $f(a) = g(a) = 0$ and let $b_n \to a$. The sequence of quotients $\frac{f(b_n)}{g(b_n)}$ may or may not converge, as both numerator and denominator go to 0. If $g'(a) \neq 0$ then $\frac{f(b_n)}{g(b_n)}$ is convergent and its limit is $\frac{f'(a)}{g'(a)}$

How would we prove it? We consider $\frac{f(b_n)}{g(b_n)} = \frac{f(b_n) - f(a)}{g(b_n) - g(a)}$ since $f(a) = g(a) = 0$. By the Cauchy MVT we can find $x_n \in]a, b_n[$ such that $\frac{f(b_n)}{g(b_n)} = \frac{f'(x_n)}{g'(x_n)}$. Thus when $b_n \to a$ we also have $x_n \to a$ and since f' and g' are assumed continuous $f'(x_n) \to f'(a)$ and $g'(x_n) \to g'(a)$. Since we assume $g'(a) \neq 0$, $\frac{f'(x_n)}{g'(x_n)} \to \frac{f'(a)}{g'(a)}$. What would happen if also $g'(a) = 0$. Then the argument would not work anymore. If however the functions f' and g' are also differentiable we could use the Cauchy MVT one more time and conclude that we can find $z_n \in]a, x_n[$ such that $\frac{f'(x_n)}{g'(x_n)} = \frac{(f')'(z_n)}{(g')'(z_n)}$. If $(g')'(a) \neq 0$ we can use the same argument as before to conclude that $\lim_{n\to\infty} \frac{f'(x_n)}{g'(x_n)} = \lim_{n\to\infty} \frac{(f')'(z_n)}{(g')'(z_n)} = \frac{(f')'(a)}{(g')'(a)}$. Hence we get $\lim_{n\to\infty} \frac{f(b_n)}{g(b_n)} = \frac{(f')'(a)}{(g')'(a)}$.

If also $(g')'(a) = 0$ the argument does not work but if the functions $(f')'$ and $(g')'$ are also differentiable we can do it again and if $((g')')'(a) \neq 0$ we get $\lim_{n\to\infty} \frac{f(b_n)}{g(b_n)} = \frac{((f')')'(a)}{((g')')'(a)}$. The functions $(f')'$ and $(g')'$ are called the *double* derivatives of f and g resp. and $((f')')'$, $((g')')'$ the triple derivatives etc. The notation for the r'th derivative is $f^{(r)}$. We have then proved the following theorem

Theorem 3.1.4 *Assume f and g are r times differentiable and assume $f(a) = f'(a) = f''(a) = \cdots = f^{(r-1)}(a) = 0 = g(a) = g'(a) = g''(a) = \cdots = g^{(r-1)}(a) = 0$ but $g^{(r)}(a) \neq 0$. If $b_n \to a$ then $\lim_{n\to\infty} \frac{f(b_n)}{g(b_n)} = \frac{f^{(r)}(a)}{g^{(r)}(a)}$*

This is a very useful result, here are a couple of examples: consider the limit as $x \to 2$ of $\frac{x-2}{x^2-4}$. Both numerator and denominator are 0 at $x = 2$ so we can't plug in directly. Applying l'Hospital's rule the limit is $\lim_{x\to 2} \frac{(x-2)'}{(x^2-4)'} = \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$. Find the limit as $x \to 0$ of $\frac{1}{x} - \frac{1}{\sin x}$. Both terms go to ∞ so we cannot directly find the limit. Rewriting we get $\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$. Now both numerator and denominator are 0 at $x = 0$. Using that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ (as we shall se later), the quotient of the derivatives is $\frac{\cos x - 1}{\sin x + x\cos x}$. Again both numerator and denominator are 0 at $x = 0$ so we cannot just plug in $x = 0$. But we can apply l'Hospital's rule again: this time we get the quotient of the derivatives is $\frac{-\sin x}{\cos x + \cos x - x \sin x}$. Now the denominator is $= 2$ at $x = 0$ and the numerator $= 0$ so the limit as $x \to 0$ is equal to 0.

Homework Problems (due Monday 11/10-2003)

Problem 1. Prove the "Nygaard MVT": Let f, g be functions continuous on $[a, b]$ and differentiable on $]a, b[$. Assume $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is a point $x_0 \in]a, b[$ such that $f'(x_0) = g'(x_0)$

Problem 2. Find the limit of $\frac{x^3 - 3x + 2}{x^3 - 4x^2 + 5x - 2}$ as $x \to 1$

Problem 3. Let f be differentiable at a point x and assume $f(x) \neq 0$. Show that the function $\frac{1}{f}$ is differentiable at $x = 0$ and $(\frac{1}{f})' = -\frac{f'(x)}{(f(x))^2}$