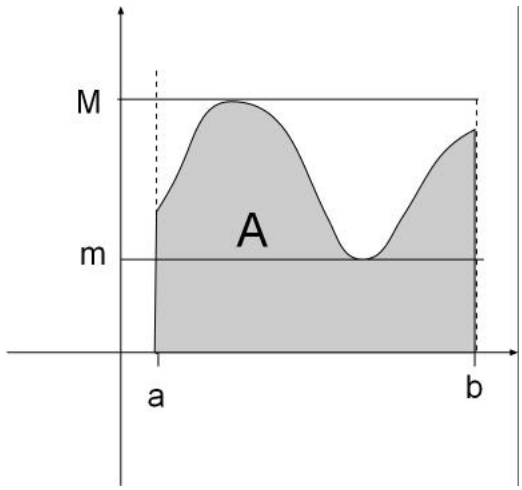

Figure 42:

4 Integration

We have already seen an example of how to compute an area using limits: in the very first lecture we constructed a sequence converging to the area of the unit disc. The method we used there, namely subdividing an interval and approximating the actual area by thinner and thinner rectangles, works very well in other cases as well.

Consider a function $f : [a, b] \to \mathbb{R}$, let us assume it is continuous and that $f(x) \geq 0$ for $x \in [a, b]$, these conditions are not really necessary for what we are doing but they make things more convenient.

Our task is to find the area under the curve $y = f(x)$ i.e. the area bounded by the x-axis, the graph $y = f(x)$ and the lines $x = a$ and $x = b$. Since f is continuous it has a maximum and a minimum in the interval $[a, b]$ say M is the maximum and m is the minimum so $m \leq f(x) \leq M$ for very $x \in [a, b]$. Clearly the area we are interested in is contained in the rectangle with base $[a, b]$ and height M and it contains the rectangle with base $[a, b]$ and height m. Hence if we let A denote the rectangle we are looking for we have $m(b - a) \leq A \leq M(b - a)$.

Now consider points in the interval, say $x_0, x_1, x_2, \ldots, x_k$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$ and consider the subintervals $J_i = [x_i, x_{i+1}]$, on each of these subintervals the function has a maximum M_i and a minimum m_i. If we look at the area bounded by $[x_i, x_{i+1}]$ and the curve an

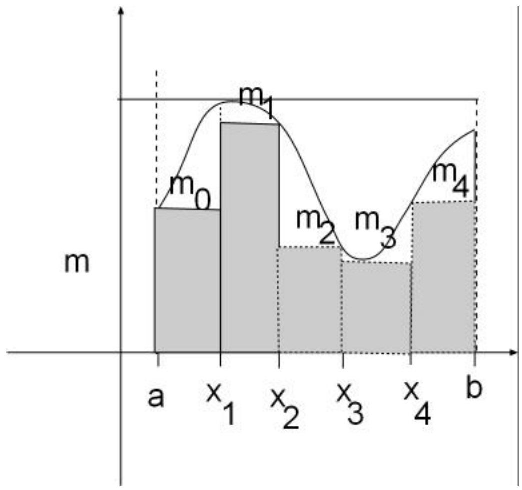

Figure 43:

call this A_i then we clearly have $m_i(x_{i+1}-x_i) \leq A_i \leq M_i(x_{i+1}-x_i$ and $A = A_0+A_1+A_2+\cdots+A_{k-1}$ and so we get the estimate $\sum\limits_{i=0} m_i(x_{i+1}-x_i) \leq A \leq \sum\limits_{i=0} M_i(x_{i+1}-x_i)$. The sum $\sum\limits_{i=0} m_i(x_{i+1}-x_i)$ is called the *lower sum* associated to the *partition* $x_0, x_1, x_2, \ldots, x_k$ and we denote it $L(f, \{x_0, x_1, \ldots, x_k\})$.

Similarly the *upper sum*, $U(f, \{x_0, x_1, \ldots, x_k\}) = \sum\limits_{i=0} M_i(x_{i+1}-x_i)$.

Remark that for any partition $\{x_0, x_1, \ldots, x_k\}$ the lower sum $L(f, \{x_0, x_1, \ldots, x_k\}) \leq M(b-a)$ and the upper sum is bounded below by $m(b-a)$. Thus the set of all lower sums is bounded above and the set of all upper sums is bounded below.

Definition 4.0.1 *The function f is said to be* integrable *(over $[a, b]$) if* $\sup\{$*all lower sums*$\} = \inf\{$*all upper sums*$\}$ *and in this case the common value is called the* integral *of f over $[a, b]$ and is denoted*

$$\int_a^b f(t)dt$$

To show that a given function is integrable, the idea is as before to take finer and finer partitions i.e. the subintervals get smaller and smaller and use the upper and lower sums to get better and better approximations to the area. To show that this works we need to make sure that the upper- and lower sums get closer and closer together so there is precisely one real

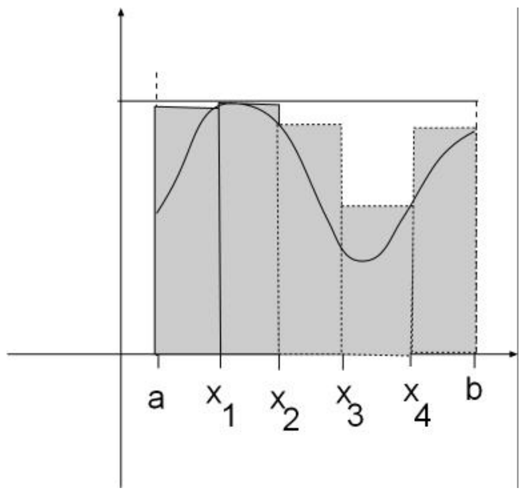

Figure 44:

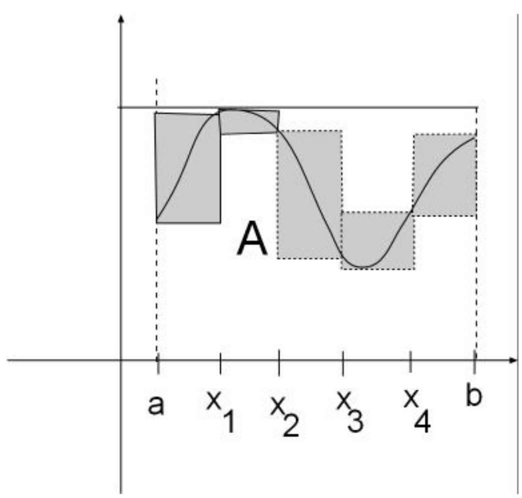

Figure 45:

number A which is between $L(f, \{z_0, z_1, \ldots, z_l\})$ and $U(f, \{z_0, z_1, \ldots, z_l\})$ for *every* partition of the interval $[a, b]$. Thus what we need to show that we can make the difference $U(f, \{x_0, x_1, \ldots, x_k\}) - L(f, \{x_0, x_1, \ldots, x_k\})$ as small as we want by taking finer and finer partitions. The difference is $= \sum_{i=0} (M_i - m_i)(x_{i+1} - x_i)$

Example 4.1 *Consider the function $f(x) = x^2$ on the interval $[0, 1]$. Consider a subdivision of the interval $0 = x_0 < x_1 < x_2 < \cdots < x_k = 1$. Since this function is increasing it is easy to find the maximum and minimum in each of the subintervals $[x_i, x_{i+1}]$ namely the maximum is x_{i+1}^2 and the minimum is x_i^2. Thus the lower sum is $\sum_{i=0} x_i^2(x_{i+1} - x_i)$ and the upper sum is $\sum_{i=0} x_{i+1}^2(x_{i+1} - x_i)$. Assume that we choose the subdivision $0 < \frac{1}{n} < \frac{2}{n} < \ldots \frac{i}{n} < \cdots < \frac{n-1}{n} < 1$. Then the difference between the upper and lower sums is $\sum_{i=0} (\frac{i+1}{n}^2 - \frac{i}{n}^2)\frac{1}{n} = \sum_{i=0} \frac{(i+1)^2}{n^3} - \frac{i^2}{n^3}$. If we write out the last sum we get $\frac{1^2}{n^3} + (\frac{2^2}{n^3} - \frac{1^2}{n^3}) + (\frac{3^2}{n^3} - \frac{2^3}{n^3}) + \cdots + (\frac{n^2}{n^3} - \frac{(n-1)^2}{n^3})$ $= \frac{n^2}{n^3} = \frac{1}{n} \to 0$. Hence the function is integrable over $[0, 1]$. Remark that this says nothing about how to compute $\int_0^1 t^2 dt$. We can compute it follows: the upper sum is $\frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + i^2 + \cdots + n^2)$. Now there is a formula for the sum of the n first squares, namely the sum is $\frac{n(n+1)(2n+1)}{6}$. (this is trivial to verify by induction, the hard part is to find the formula). It follows that the upper sum is $\frac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \to \frac{1}{3}$. Thus the area bounded by the interval $[0, 1]$ and the graph of $y = x^2$ equals $\frac{1}{3}$. But if we did not know the formula for the sum of the n first squares we would have had a hard time computing the integral.*

Theorem 4.0.5 *Let $\{x_0, x_1, x_2, \ldots, x_k\}$ and $\{z_0, z_1, \ldots, z_l\}$ be any two partitions of $[a, b]$. Then $L(f, \{x_0, x_1, \ldots, x_k\}) \leq U(f, \{z_0, z_1, \ldots, z_l\})$*

This result is a little surprising: it is clear that $m_i \leq M_i$ for any subinterval J_i so we do have $L(f, \{x_0, x_1, \ldots, x_k\}) = \sum_{i=0} m_i(x_{i+1} - x_i) \leq$

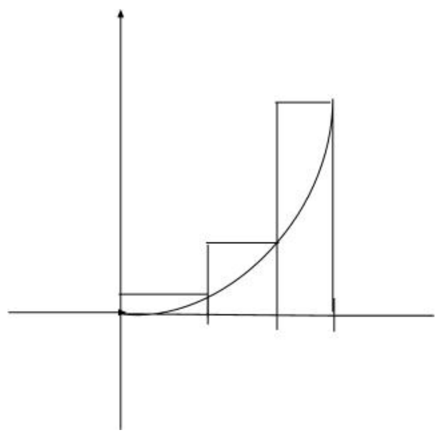

Figure 46:

$\sum_{i=0} M_i(x_{i+1} - x_i) = U(f, \{x_0, x_1, \ldots, x_k\})$. Thus for the same partition the upper sum is always $\geq$ the lower sum. The theorem says that this is also true when we take two different partitions. We cannot directly compare maximum and minimum over two different subintervals $[z_j, z_{j+1}]$ and $[x_i, x_{i+1}]$. The trick is to take the union of the two sets and make a partition out of the union. Consider the union $\{x_0, x_1, \ldots, x_k, z_0, z_1, \ldots, z_l\}$ and order the set and use the ordered set as a new set of division points. In the example the union of the two partitions $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ and $\{z_0, z_1, z_2, z_3, z_4, z_5, z_6\}$ is the partition $\{x_0, z_1, x_1, z_2, x_2, z_3, x_3, z_4, x_4, z_5, x_5\}$. Now look at the minimum over the interval $[x_i, x_{i+1}]$. If one of the z_j's are in this interval then it gets split up in the union partition: $[x_i, x_{i+1}] = [x_i, z_j] \bigcup [z_j, x_{i+1}]$. Then the minimum of f on the interval $[x_i, x_{i+1}]$ has to be $\leq$ to the minimums on both $[x_i, z_j]$ and $[z_j, x_{i+1}]$, if we denote these minima $m_{i,j}$ and $m_{i+1,j}$ then $m_i \leq m_{i,j}$ and $m_{i+1,j}$ and so $m_i(x_{i+1} - x_i) = m_i(z_j - x_i) + m_i(x_{i+1} - z_j) \leq m_{i,j}(z_j - x_i) + m_{i+1,j}(x_{i+1} - z_j)$ Summing over all the subintervals we get $L(f, \{x_0, x_1, \ldots, x_k\}) \leq L(f, \{x_0, x_1, \ldots, x_k, z_0, z_1, \ldots, z_l\})$. Next we look at the upper sums. Consider an interval $[z_j, z_{j+1}]$ and suppose that we have an x_i in the interval. Then the maximum over the interval $[z_j, z_{j+1}] \geq$ the maxima over the intervals $[z_j, x_i]$ and $[x_i, z_{j+1}]$. Hence $M_j(z_{j+1} - z_j) = M_j(x_i - z_j) + M_j(z_{j+1} - x_i) \geq M_{ji}(x_i - z_j) + M_{j+1,i}(z_{j+1} - x_i)$. Summing over all the subintervals we get $U(f, \{x_0, x_1, \ldots, x_k, z_0, z_1, \ldots, z_l\}) \leq U(f, \{z_0, z_1, \ldots, z_l\})$ It follows that

$$U(f, \{x_0, x_1, \ldots, x_k, z_0, z_1, \ldots, z_l\}) \leq U(f, \{z_0, z_1, \ldots, z_k\})$$

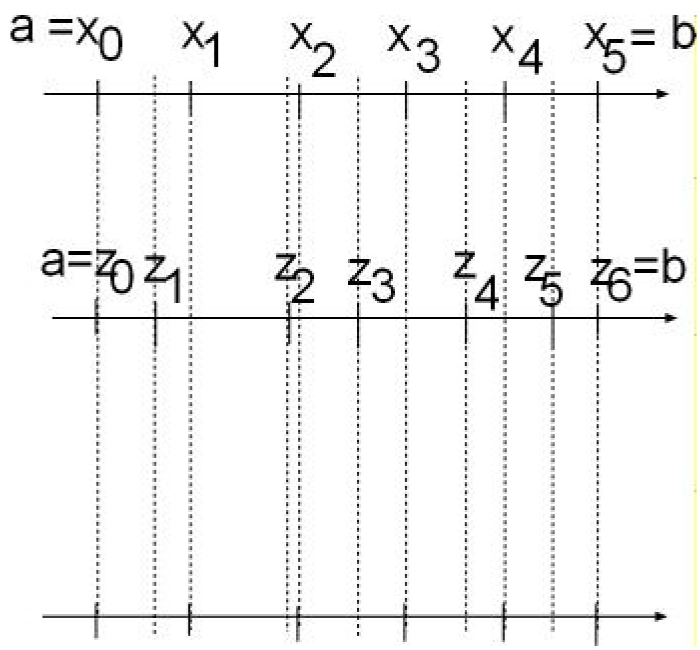

Figure 47:

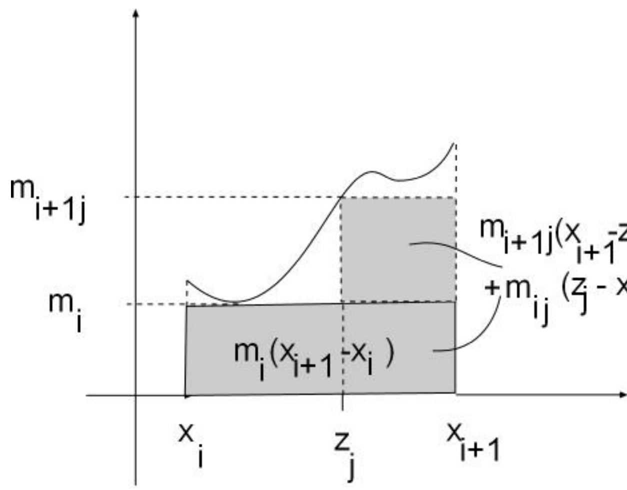

Figure 48:

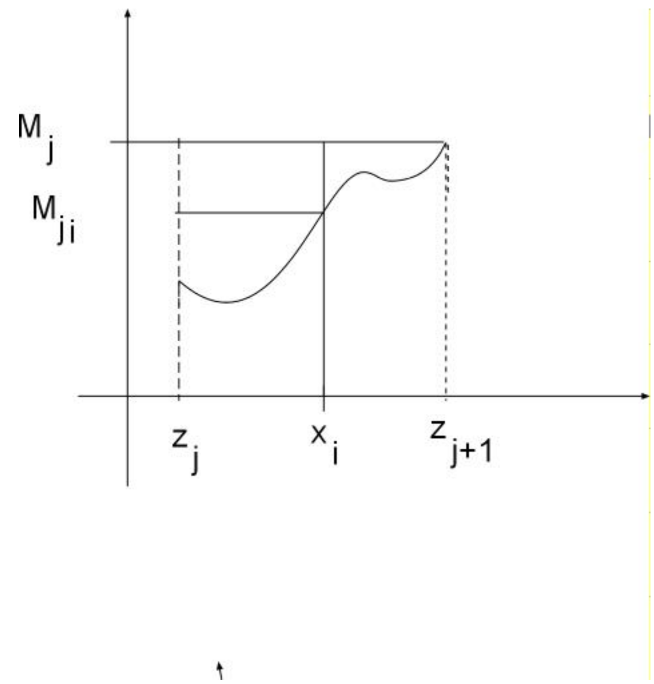

Figure 49:

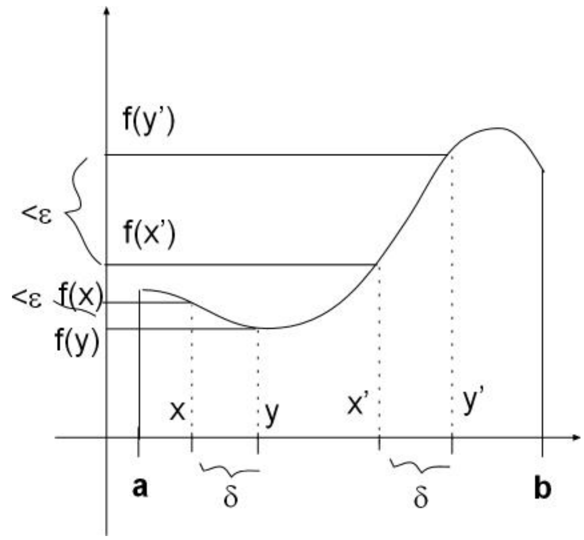

Figure 50:

Putting these two inequalities together we get $L(f, \{x_0, x_1, \ldots, x_k\}) \leq U(f, \{z_0, z_1, \ldots, z_l\})$

We are now going to show that our continuous function is integrable. Thus we have to show that we can make the difference between the upper and lower sum as small as we want. The exact theorem is the following:

Theorem 4.0.6 *Let $\varepsilon > 0$ be given, then there exists a partition $\{x_0, x_1, \ldots, x_k\}$ of the interval $[a, b]$ such that $U(f, \{x_0, x_1, x_2, \ldots, x_k\}) - L(f, \{x_0, x_1, x_2, \ldots, x_k\}) < \varepsilon$*

We shall first show that a continuous function on a closed, bounded interval $[a, b]$ has the following remarkable property: given any $\varepsilon > 0$ there is another number $\delta > 0$ such that for any two points x, y in $[a, b]$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. This property is called *uniform* continuity. The reason we call it remarkable is that from the figures that one can draw, it does not look like it is true (see fig. 50). It is, however true and here is the argument: Assume it was not true, that would mean that there would be some ε for which we could not find such a δ. What that means is that no matter which δ we take we can always find a pair of points x and y in $[a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Now we take $\delta = \frac{1}{n}$. Then we can find two points x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Consider the sequence $\{x_n\} \subset [a, b]$. Since $[a, b]$ is

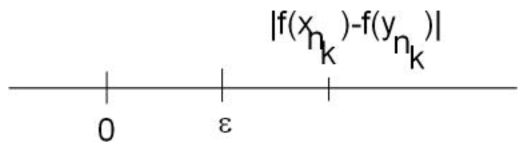

Figure 51:

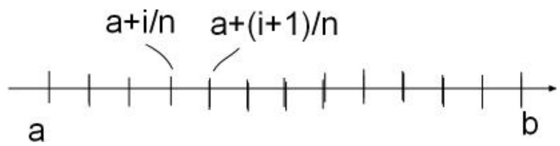

Figure 52:

compact we can find a convergent subsequence $x_{n_k} \to x$ say. Consider now the sequence y_{n_k}. We claim that also $y_{n_k} \to x$. We have $|x - y_{n_k}| = |x - x_{n_k} + x_{n_k} - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| < |x - x_{n_k}| + \frac{1}{n_k}$. Both of the two terms can be made arbitrarily small so we can make $|x - y_{n_k}|$ arbitrarily small. Since f is continuous both $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. Hence $|f(x_{n_k}) - f(y_{n_k})| \to |f(x) - f(x)| = 0$, on the other hand $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ so it cannot converge to 0. This contradiction proves the uniform continuity.

Proposition 4.0.1 *A continuous function $f : [a, b] \to \mathbb{R}$ on a closed, bounded interval is uniformly continuous*

Now we can easily prove that a continuous function f on a closed interval $[a, b]$ is integrable. Again we have to show that we can find a partition $\{x_0, x_1, \ldots, x_k\}$ such that $U(f, \{x_0, x_1, \ldots, x_k\}) - L(f, \{x_0, x_1, \ldots, x_k\}) < \varepsilon$

Consider $\frac{\varepsilon}{b-a}$. Using the uniform continuity we can find δ such that for any two points $x, y \in [a, b]$, if $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Now take a partition so fine that the length of all the subintervals $[x_i, x_{i+1}]$ are $< \delta$ (for instance if we first choose n such that $\frac{1}{n} < \delta$ we can choose the partition where $x_i = a + \frac{i}{n}$ for $i \leq n(b-a)$). Now M_i and m_i are two values of the function on the interval $[x_i, x_{i+1}]$. Since the length of this interval is $\frac{1}{n} < \delta$ the distance between any to values of the function on this interval is

$< \dfrac{\varepsilon}{b-a}$ and so $M_i - m_i < \dfrac{\varepsilon}{b-a}$. It follows that for this partition we have $U(f, \{x_o, x_1, \ldots, x_k\}) - L(f, \{x_0, x_1, \ldots, x_k\}) = \sum\limits_{i=0} (M_i - m_i)(x_{i+1} - x_i) <$
$\sum\limits_{i=0} \dfrac{\varepsilon}{b-a}(x_{i+1} - x_i) = \dfrac{\varepsilon}{b-a} \sum\limits_{i=0} (x_{i+1} - x_i) = \dfrac{\varepsilon}{b-a}(b-a) = \varepsilon$

Homework Problems (due Monday 11/17-2003)

Problem 1. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 \text{ if } x \text{ is a rational number} \\ 1 \text{ ix } x \text{ is not a rational number} \end{cases}$ Show that this function is not integrable (Show that all the upper sums are $= 1$ and all the lower sums are $= 0$)

Problem 2. Use induction to prove that for all natural numbers n

$$1^2 + 2^2 + 3^2 + \cdots + i^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 3. Show that for any $b > 0$ the function $f : [0, b] \to \mathbb{R}$ is integrable over $[0, b]$ and compute the integral

$$\int_0^b t^2 dt$$