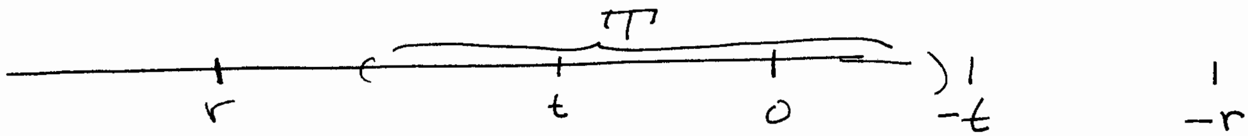
Homework Solutions

Problem 1

If $T \subset \mathbb{R}$ is bounded below there is a real number r such that $r \leq t$ for every t in T



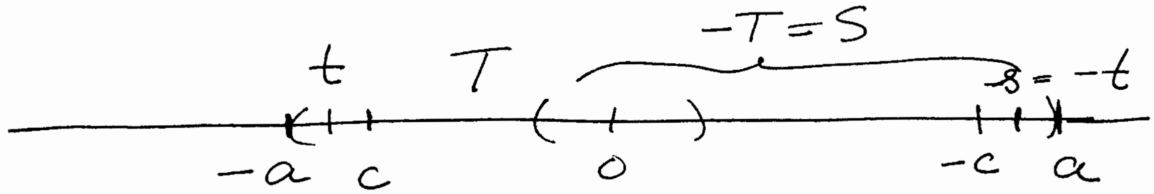
If $r \leq t$ we have $-r \geq -t$ and so $-r \geq s$ for $s = -t$ i.e. $-r$ is an upper bound for $S = -T$

Problem 2

Let $a = \sup S$, where $S = -T$ (this exists because we have shown in problem 1 that S is bounded above and so S has a lowest upper bound, $\sup S$).

This means that $a \geq s$ for every $s \in S$, hence $-a \leq -s$ for every $s \in S = -T$. But $s = -t$ so $-s = t$. Thus $-a \leq t$ for every $t \in T$.

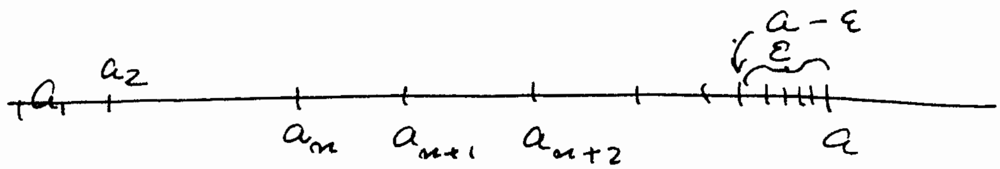
This shows that $-a$ is a lower bound for T.
We have to show that it is a largest lower bound. Thus we have to show that if $c > -a$ then c is not a lower bound, i.e. that there is some $t \in T$ such that $t < c$



Now $c > -a$ means that ~~there~~ $-c < a$ and since a is the lowest upper bound for S $-c$ cannot be an upper bound for S. Hence there must be a $s \in S$ such that $s > -c$, but $s = -t$ for some $t \in T$, and hence $t < c$ i.e. c is not a lower bound.

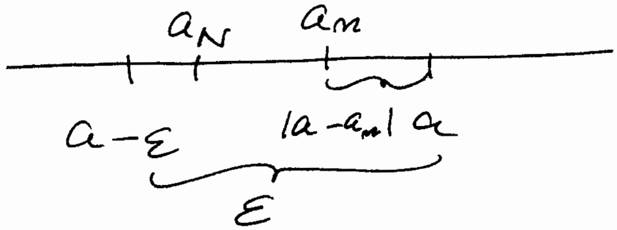
Problem 3

Let $S = \{a_1, a_2, a_3, \ldots\} = \{a_n\}$.
We are given that there exist $A \in \mathbb{R}$ such that $a_n \le A$ for all n, i.e. A is an upper bound for S.
Thus S is bounded above and hence S has a lowest upper bound $a = \sup S$.
We shall show that $a_n \longrightarrow a$.
Thus we have to show that for every $\varepsilon > 0$ we can find N (depending on ε) such that when $n > N$
$|a - a_n| < \varepsilon$



Consider $a - \varepsilon$. Since $\varepsilon > 0$, $a - \varepsilon < a$ and since a is the lowest upper bound $a - \varepsilon$ cannot be an upper bound.
Hence we can find N such that $a_N > a - \varepsilon$, i.e. the N'th term in the sequence is $> a - \varepsilon$. If there was no such N all the terms in the sequence would be $\le a - \varepsilon$ and so $a - \varepsilon$ would be an upper bound for $S = \{a_1, a_2, a_3, \ldots, a_N, \ldots, a_n, \ldots\}$ but it is not.

Now if $n > N$ we have
$a_n \geq a_N$ because the sequence
is increasing
$\leq a_N \leq a_{N+1} \leq a_{N+2} \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$
Hence
$$a - \varepsilon < a_N \leq a_n \leq a$$

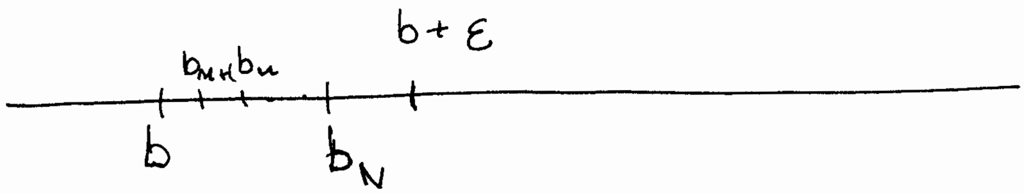


but then $|a - a_n| < \varepsilon$, this shows
that $a_n \longrightarrow a$.

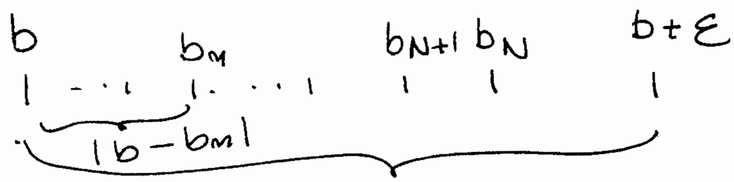
Problem 4

Let $T = \{b_1, b_2, b_3, \ldots, b_n, \ldots \}$
Since there is a B such that
$B \leq b_n$ for every $b_n \in T$, the
set T is bounded below. Hence
by Problem 2, T has a largest
lower bound, $b = \inf T$.
We shall show that $b_n \longrightarrow b$
Let $\varepsilon > 0$ be given, we have to find
N such that for every $n > N$
$|b - b_n| < \varepsilon$.

Consider $b+\varepsilon$, since this is $> b$ and b is the largest lower bound $b+\varepsilon$ cannot be a lower bound. Hence there must be $b_N \in T$ such that $b_N < b+\varepsilon$



Now again if $n > N$, $b_n \leq b_N$



Hence $|b-b_n| \leq |b-b_N| < \varepsilon$

so $b_n \longrightarrow b$.