1 A One Period Market Model

We consider a simple model of a market. The market consists of a population of I traders, trading in a market with N securities. There are two significant times: a time $t = 0$ and a time $t = T$ later. The state of the world at time $t = T$ is unknown at time $t = 0$ but it is described by a set of possible states $\{\omega_1, \omega_2, \omega_3, \dots, \omega_K\} = \Omega$, so we know at time 0 that at time T the world will be in one of these states but we do not know which of them it will be. Each security pays off a certain amount at time T, the amount being dependent on which state is prevailing at time T. Thus the possible pay-offs of security j is described by a vector $\begin{pmatrix} d_j(\omega_1) \\ d_j(\omega_2) \\ \vdots \\ d_j(\omega_K) \end{pmatrix} \in \mathbb{R}^K$. We collect these data in the *pay-off matrix*

$$D = \begin{pmatrix} d_1(\omega) & d_2(\omega_1) & \dots & d_N(\omega_1) \\ d_1(\omega_2) & d_2(\omega_2) & \dots & d_N(\omega_2) \\ d_1(\omega_3) & d_2(\omega_3) & \dots & d_N(\omega_3) \\ \vdots & \vdots & \vdots & \vdots \\ d_1(\omega_K) & d_2(\omega_K) & \dots & d_N(\omega_K) \end{pmatrix}$$

Thus the pay-off matrix is a $K \times N$ matrix.

A portfolio is a vector $\underline{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix} \in \mathbb{R}^N$ where each θ_i is the number of shares of the i'th security. The pay-off vector of a portfolio $\underline{\theta}$, $\begin{pmatrix} \underline{\theta}(\omega_1) \\ \underline{\theta}(\omega_2) \\ \vdots \\ \underline{\theta}(\omega_K) \end{pmatrix} \in \mathbb{R}^K$ is given by $D \cdot \underline{\theta}$, so the pay-off matrix defines a linear map from the space of portfolios, $\mathbb{R}^N$ to the space of pay-offs, $\mathbb{R}^K$. The image of this linear map is a subspace of $\mathbb{R}^K$. This subspace consists of all pay-offs that can be realized by portfolios of the N securities.

At time $t = 0$ trading takes place at prices $\underline{p} = (p_1, p_2, \dots, p_N)$, hence the price of putting together a portfolio $\underline{\theta}$ is $\underline{p} \cdot \underline{\theta} = p_1\theta_1 + p_2\theta_2 + \dots + p_N\theta_N$.

Consider two portfolios $\underline{\theta}$ and $\underline{\psi}$, and assume $D \cdot \underline{\theta} = D \cdot \underline{\psi}$. If $\underline{p} \cdot \underline{\theta} \neq \underline{p} \cdot \underline{\psi}$ we have an *arbitrage* opportunity: if $\underline{p} \cdot \underline{\theta} > \underline{p} \cdot \underline{\psi}$ we sell the portfolio $\underline{\theta} - \underline{\psi}$.

This will generate positive cash flow $\underline{p} \cdot (\underline{\theta} - \underline{\psi})$ and at time $t = T$ will pay-off $D \cdot (\underline{\theta} - \underline{\psi}) = \underline{0}$ and so no matter what happens we get to keep the profit from the trade, i.e. we have a guaranteed profit.

Example 1.1 *Consider a market with* 7 *securities and* 5 *states* $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \Omega$. *Assume the pay-off matrix is given by*

$$D = \begin{pmatrix} 5 & 7 & 0 & 10 & 1 & 6 & 0 \\ 5 & 4 & 0 & 5 & 4 & 6 & -2 \\ 5 & 6 & 0 & 7 & 1 & 6 & 5 \\ 5 & 3 & 0 & 1 & 4 & 6 & 6 \\ 5 & 10 & 3 & 4 & 5 & -1 & -9 \end{pmatrix}$$

Remark that the pay-off of the first security (the first column) is independent of the state, this is called a risk free security. *Assume the initial price vector is* $(4, 10, 1, 6, 2, 5, 3)$. *To discover arbitrage opportunities we shall first compute the kernel of the linear map* $D : \mathbb{R}^7 \to \mathbb{R}^5$.

The reduced row echelon form of D *is* $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1.2 & .73 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & -2.3 & -10.78 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1.67 \end{pmatrix}$ *This shows that the image of* D *has dimension* 5 *i.e.* D *is onto. Thus the kernel has dimension* 2 *and we see that* $\underline{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_7 \end{pmatrix}$ *is in the kernel if and only*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1.2 & .73 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & -2.3 & -10.78 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1.67 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_7 \end{pmatrix} = \begin{pmatrix} \theta_1 + 1.2\theta_6 + .73\theta_7 \\ \theta_2 + 4\theta_7 \\ \theta_3 - 2.3\theta_6 - 10.78\theta_7 \\ \theta_4 - 3\theta_7 \\ \theta_5 - 1.67\theta_7 \end{pmatrix} = \underline{0}.$$

This shows that $\ker D$ *is the subspace of vectors of the form*

$$\begin{pmatrix} -1.2 \\ 0 \\ 2.3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \theta_6 + \begin{pmatrix} -.73 \\ -4 \\ 10.78 \\ 3 \\ 1.67 \\ 0 \\ 1 \end{pmatrix} \theta_7$$

Hence any portfolio which is a linear combination of these two pays off 0 *in any state at time* T.

The initial price of the portfolio $\begin{pmatrix} -1.2 \\ 0 \\ 2.3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ *is* $(4, 10, 1, 6, 2, 5, 3) \cdot \begin{pmatrix} -1.2 \\ 0 \\ 2.3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$ 2.5. *Hence this is an arbitrage portfolio. By selling this portfolio we make a guaranteed profit of* 2.5.

A security is completely described by its initial price and the pay-off vector i.e. we can view a security as a vector in $\mathbb{R}^{1+K}$. A portfolio $\underline{\theta} \in \ker D$ will be an arbitrage portfolio unless its initial price $\underline{p} \cdot \underline{\theta} = 0$.

Consider now a exogenous security, with pay-off vector $\underline{d} = \begin{pmatrix} d(\omega_1) \\ d(\omega_2) \\ \vdots \\ (\omega_K) \end{pmatrix} \in \mathbb{R}^K$. We say that the security can be *replicated* if there is a portfolio $\underline{\theta}$ such that $D \cdot \underline{\theta} = \underline{d}$.

Proposition 1.0.1 *Any security can be replicated if and only if* D *has column rank*$=K$

Proof: To say that any security can be replicated is the same as saying that for any $\underline{d} \in \mathbb{R}^K$ there exists $\underline{\theta} \in \mathbb{R}^N$ such that $D \cdot \underline{\theta} = \underline{d}$. This is the same as saying that D is onto i.e. that Im $D = \mathbb{R}^K$. But Im D is spanned by the columns of D, indeed $D \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix} = \theta_1 \begin{pmatrix} d_1(\omega_1) \\ d_1(\omega_2) \\ \vdots \\ d_1(\omega_K) \end{pmatrix} + \theta_2 \begin{pmatrix} d_2(\omega_1) \\ d_2(\omega_2) \\ \vdots \\ d_2(\omega_K) \end{pmatrix} + \cdots + \theta_N \begin{pmatrix} d_N(\omega_1) \\ d_N(\omega_2) \\ \vdots \\ d_N(\omega_K) \end{pmatrix}$ which shows that $D\underline{\theta}$ is a linear combination of the columns of D. Thus D is onto if and only if the columns of D span $\mathbb{R}^K$ i.e. the column rank is K.

Suppose now that the market is arbitrage free i.e. there are no arbitrage portfolios. Then a security which can be replicated, say by a portfolio $\underline{\theta}$ must have initial price equal to the price of the replicating portfolio. This is the underlying principle in arbitrage pricing: in an arbitrage free market two portfolios with the same pay-off vectors must have the same initial price.

Example 1.2 *Consider a market with two securities and two states. We assume the pay-off matrix is given by* $D = \begin{pmatrix} d & d(\omega_1) \\ d & d(\omega_2) \end{pmatrix}$. *Thus the securities are a risk free security (a bond) and a risky security (a stock). The columns are linearly dependent if and only if* $d \neq 0$ *and* $d(\omega_1) \neq d(\omega_2)$ *which we will assume it is (otherwise both securities would be risk free). We assume that* $d(\omega_1) < d(\omega_2)$. *Let* K *be a number such that* $d(\omega_1) < K < d(\omega_2)$. *Consider now a security with pay-off vector* $\begin{pmatrix} c(\omega_1) \\ c(\omega_2) \end{pmatrix} = \begin{pmatrix} 0 \\ d(\omega_2) - K \end{pmatrix}$. *This is a call option on the stock with strike* K. *Since* D *has column rank* $= 2$ *it can be replicated indeed we have* $\begin{pmatrix} 0 \\ d(\omega_2) - K \end{pmatrix} = \Delta \begin{pmatrix} d(\omega_1) \\ d(\omega_2) \end{pmatrix} + B \begin{pmatrix} d \\ d \end{pmatrix}$ *where* $\Delta = \dfrac{d(\omega_2) - K}{d(\omega_2) - d(\omega_1)}$ *and* $B = -\dfrac{1}{d} \dfrac{d(\omega_2) - K}{d(\omega_2) - d(\omega_1)} d(\omega_1)$. *If the initial prices are* $(1, S_0)$ *the price of the call must be* $C = \Delta S_0 + B$ *in order to avoid arbitrage.*

We shall next assume that each trader associates probabilities to the states i.e. to each state $\omega \in \Omega$ the trader has associated a number $P(\omega)$ such that $0 \le P(\omega) \le 1$ and $\sum\limits_{\omega\in\Omega} P(\omega) = 1$. Having this *probability measure* allows us to incorporate expectations and volatility into the model.

It is more convenient to consider returns than the actual pay-offs, thus if the pay-off of Sec_n in state ω_k is $d_n(\omega_k)$ and the initial price is p_n, the return in state ω_k is $r_n(\omega_k) = \dfrac{d_n(\omega_k)}{p_n}$. The return on a portfolio $\underline{\theta}$ in state ω_k is then given by $r(\underline{\theta})(\omega_k) = \dfrac{\sum\limits_n d_n(\omega_k)\theta_n}{\underline{p}\cdot\underline{\theta}}$.

Definition 1.0.1 *Let $X : \Omega \to \mathbb{R}$ be a function on the state space. The* expectation *of X (or the expected value of X) is the weighted sum $\sum\limits_k P(\omega_k)X(\omega_k)$. The* variance *of X is defined by $Var(X) = \mathbb{E}((X-\mathbb{E}(X))^2) = \sum\limits_k P(\omega_k)(X(\omega_k)-\mathbb{E}(X))^2$*

Remark that we can write $(X(\omega_k) - \mathbb{E}(X))^2 = X(\omega_k)^2 + \mathbb{E}(X)^2 - 2\mathbb{E}(X)X(\omega_k)$. Hence $Var(X) = \sum\limits_k P(\omega_k)(X(\omega_k)^2+\mathbb{E}(X)^2-2\mathbb{E}(X)X(\omega_k)) = \sum\limits_k P(\omega_k)X(\omega_k)^2+\sum\limits_k P(\omega_k)\mathbb{E}(X)^2-\sum\limits_k P(\omega_k)\mathbb{E}(X)X(\omega_k) = \mathbb{E}(X^2)+\mathbb{E}(X)^2\sum\limits_k P(\omega_k)-2\mathbb{E}(X)\sum\limits_k P(\omega_k)X(\omega_k)$ Since $\sum\limits_k P(\omega_k) = 1$ and $\sum\limits_k P(\omega_k)X(\omega_k) = \mathbb{E}(X)$ the last expression becomes $\mathbb{E}(X^2) + \mathbb{E}(X)^2 - 2\mathbb{E}(X)\mathbb{E}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

The *expected return* on the portfolio $\underline{\theta}$, $\mathbb{E}(r(\underline{\theta}))$ is then the weighted sum $\sum\limits_k P(\omega_k)r(\underline{\theta})(\omega_k)$ and the variance on the return is $Var(r(\underline{\theta})) = \sum\limits_k P(\omega_k)r(\underline{\theta})(\omega_k)^2-(\sum\limits_k P(\omega_k)r(\underline{\theta})(\omega_k))^2$

Definition 1.0.2 *Let $X, Y : \Omega \to \mathbb{R}$ be functions. The* covariance *between X and Y is given by $Cov(X,Y) = \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))$*

We have $(X-\mathbb{E}(X))(Y-\mathbb{E}(Y)) = XY - X\mathbb{E}(Y) - Y\mathbb{E}(X) + \mathbb{E}(X)\mathbb{E}(Y)$ and so $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Remark that $Cov(X,X) = Var(X)$. The correlation coefficient $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

Consider the returns on securities n and m say r_n and r_m (they are functions of the state). Then we have $Cov(r_n, r_m) = \sum\limits_k P(\omega_k)r_n(\omega_k)r_m(\omega_k) -$

$\sum_k P(\omega_k)r_n(\omega_k)\sum_k P(\omega_k)r_m(\omega_k)$ We can then form *covariance matrix*

$$C = \begin{pmatrix} Var(r_1) & Cov(r_1,r_2) & Cov(r_1,r_3) & \dots & Cov(r_1,r_N) \\ Cov(r_2,r_1) & Var(r_2,r_2) & Cov(r_2,r_3) & \dots & Cov(r_2,r_N) \\ Cov(r_3,r_1) & Cov(r_3,r_2) & Var(r_3) & \dots & Cov(r_3,r_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Cov(r_N,r_1) & Cov(r_n,r_2) & Cov(r_n,r_3) & \dots & Var(r_N) \end{pmatrix}$$

Remark that since $Cov(r_n, r_m)$ is clearly equal to $Cov(r_m, r_n)$ the covariance matrix is symmetric i.e. ${}^tC = C$. Also C is an $N \times N$ matrix. We shall now see how we can compute C from the pay-off matrix D, the initial price vector $\underline{p} = (p_1, p_2, \dots, p_N)$ and the probabilities $\underline{prob} = \begin{pmatrix} P(\omega_1) \\ P(\omega_2) \\ \vdots \\ P(\omega_K) \end{pmatrix}$. First we need to compute the returns matrix

$$R = \begin{pmatrix} r_1(\omega_1) & r_2(\omega_1) & r_3(\omega_1) & \dots & r_N(\omega_1) \\ r_1(\omega_2) & r_2(\omega_2) & r_3(\omega_2) & \dots & r_N(\omega_2) \\ r_1(\omega_3) & r_2(\omega_3) & r_3(\omega_3) & \dots & r_N(\omega_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_1(\omega_K) & r_2(\omega_K) & r_3(\omega_K) & \dots & r_N(\omega_K) \end{pmatrix}$$

$$= \begin{pmatrix} d_1(\omega_1)/p_1 & d_2(\omega_1)/p_2 & d_3(\omega_1)/p_3 & \dots & d_N(\omega_1)/p_N \\ d_1(\omega_2)/p_1 & d_2(\omega_2)/p_2 & d_3(\omega_2)/p_3 & \dots & d_N(\omega_2)/p_2 \\ d_1(\omega_3)/p_1 & d_2(\omega_3)/p_2 & d_3(\omega_3)/p_3 & \dots & d_N(\omega_3)/p_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_1(\omega_K)/p_1 & d_2(\omega_K)/p_2 & d_3(\omega_K)/p3 & \dots & d_N(\omega_K)/p_N \end{pmatrix}$$

$$= D \cdot \begin{pmatrix} \frac{1}{p_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{p_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{p_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{p_N} \end{pmatrix}$$

The expected return on Sec_n is $\mathbb{E}(r_n) = \sum_k P(\omega_k)r_n(\omega_k)$ and so the vector of expected returns

$$\begin{pmatrix} \mathbb{E}(r_1) \\ \mathbb{E}(r_2) \\ \vdots \\ \mathbb{E}(r_N) \end{pmatrix} = {}^tR \cdot \underline{prob} = \begin{pmatrix} r_1(\omega_1) & r_1(\omega_2) & \dots & r_1(\omega_K) \\ r_2(\omega_1) & r_2(\omega_2) & \dots & r_2(\omega_K) \\ \vdots & \vdots & \vdots & \vdots \\ r_N(\omega_1) & r_N(\omega_2) & \dots & r_N(\omega_K) \end{pmatrix} \begin{pmatrix} P(\omega_1) \\ P(\omega_2) \\ \vdots \\ P(\omega_K) \end{pmatrix}$$

Let $\mathbb{P}$ denote the $K \times K$ diagonal matrix $\begin{pmatrix} P(\omega_1) & 0 & 0 & \dots & 0 \\ 0 & P(\omega_2) & 0 & \dots & 0 \\ 0 & 0 & P(\omega_3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P(\omega_K) \end{pmatrix}$

We want to find an expression for the covariance matrix in terms of the returns matrix and the matrices above: the covariance matrix is $C =$

$$\begin{pmatrix} \mathbb{E}(r_1^2) - \mathbb{E}(r_1)^2 & \mathbb{E}(r_1 r_2) - \mathbb{E}(r_1)\mathbb{E}(r_2) & \dots & \mathbb{E}(r_1 r_N) - \mathbb{E}(r_1)\mathbb{E}(r_N) \\ \mathbb{E}(r_2 r_1) - \mathbb{E}(r_2)\mathbb{E}(r_1) & \mathbb{E}(r_2^2) - \mathbb{E}(r_2)^2 & \dots & \mathbb{E}(r_2 r_N) - \mathbb{E}(r_2)\mathbb{E}(r_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(r_N r_1) - \mathbb{E}(r_N)\mathbb{E}(r_1) & \mathbb{E}(r_N r_2) - \mathbb{E}(r_N)\mathbb{E}(r_2) & \dots & \mathbb{E}(r_N^2) - \mathbb{E}(r_N)^2 \end{pmatrix} =$$

$$\begin{pmatrix} \mathbb{E}(r_1^2) & \mathbb{E}(r_1 r_2) & \dots & \mathbb{E}(r_1 r_N) \\ \mathbb{E}(r_2 r_1) & \mathbb{E}(r_2^2) & \dots & \mathbb{E}(r_2 r_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(r_N r_1) & \mathbb{E}(r_N r_2) & \dots & \mathbb{E}(r_N^2) \end{pmatrix} - \begin{pmatrix} \mathbb{E}(r_1)^2 & \mathbb{E}(r_1)\mathbb{E}(r_2) & \dots & \mathbb{E}(r_1)\mathbb{E}(r_N) \\ \mathbb{E}(r_2)\mathbb{E}(r_1) & \mathbb{E}(r_2)^2 & \dots & \mathbb{E}(r_2)\mathbb{E}(r_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(r_N)\mathbb{E}(r_1) & \mathbb{E}(r_N)\mathbb{E}(r_2) & \dots & \mathbb{E}(r_N)^2 \end{pmatrix}$$

Now $^tR \cdot \mathbb{P} \cdot R = \begin{pmatrix} r_1(\omega_1) & r_1(\omega_2) & \dots & r_1(\omega_K) \\ r_2(\omega_1) & r_2(\omega_2) & \dots & r_2(\omega_K) \\ \vdots & \vdots & \vdots & \vdots \\ r_N(\omega_1) & r_N(\omega_2) & \dots & r_N(\omega_K) \end{pmatrix} \cdot \begin{pmatrix} P(\omega_1) & 0 & \dots & 0 \\ 0 & P(\omega_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P(\omega_K) \end{pmatrix} \cdot$

$$\begin{pmatrix} r_1(\omega_1) & r_2(\omega_1) & \dots & r_N(\omega_1) \\ r_1(\omega_2) & r_2(\omega_2) & \dots & r_N(\omega_2) \\ \vdots & \vdots & \vdots & \vdots \\ r_1(\omega_K) & r_2(\omega_K) & \dots & r_N(\omega_K) \end{pmatrix}$$

$$= \begin{pmatrix} P(\omega_1)r_1(\omega_1) & P(\omega_2)r_1(\omega_2) & \dots & P(\omega_K)r_1(\omega_K) \\ P(\omega_1)r_2(\omega_1) & P(\omega_2)r_2(\omega_2) & \dots & P(\omega_K)r_2(\omega_K) \\ \vdots & \vdots & \vdots & \vdots \\ P(\omega_1)r_N(\omega_1) & P(\omega_2)r_N(\omega_2) & \dots & P(\omega_K)r_N(\omega_K) \end{pmatrix} \cdot \begin{pmatrix} r_1(\omega_1) & r_2(\omega_1) & \dots & r_N(\omega_1) \\ r_1(\omega_2) & r_2(\omega_2) & \dots & r_N(\omega_2) \\ \vdots & \vdots & \vdots & \vdots \\ r_1(\omega_K) & r_2(\omega_K) & \dots & r_N(\omega_K) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_k P(\omega_k) r_1(\omega_k)^2 & \sum_k P(\omega_k) r_1(\omega_k) r_2(\omega_k) & \ldots & \sum_k P(\omega_k) r_1(\omega_k) r_N(\omega_k) \\ \sum_k P(\omega_k) r_2(\omega_k) r_1(\omega_k) & \sum_k P(\omega_k) r_2(\omega_k)^2 & \ldots & \sum_k P(\omega_k) r_2(\omega_k) r_N(\omega_k) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_k P(\omega_k) r_N(\omega_k) r_1(\omega_k) & \sum_k P(\omega_k) r_N(\omega_k) r_2(\omega_k) & \ldots & \sum_k P(\omega_k) r_N(\omega_k)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{E}(r_1^2) & \mathbb{E}(r_1 r_2) & \ldots & \mathbb{E}(r_1 r_N) \\ \mathbb{E}(r_2 r_1) & \mathbb{E}(r_2^2) & \ldots & \mathbb{E}(r_2 r_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(r_N r_1) & \mathbb{E}(r_N r_2) & \ldots & \mathbb{E}(r_N^2) \end{pmatrix}$$

Next $\begin{pmatrix} \mathbb{E}(r_1) \\ \mathbb{E}(r_2) \\ \vdots \\ \mathbb{E}(r_N) \end{pmatrix} \cdot (\mathbb{E}(r_1), \mathbb{E}(r_2), \ldots, \mathbb{E}(r_N)) = \begin{pmatrix} \mathbb{E}(r_1)^2 & \mathbb{E}(r_1)\mathbb{E}(r_2) & \ldots & \mathbb{E}(r_1)\mathbb{E}(r_N) \\ \mathbb{E}(r_2)\mathbb{E}(r_1) & \mathbb{E}(r_2)^2 & \ldots & \mathbb{E}(r_2)\mathbb{E}(r_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(r_N)\mathbb{E}(r_1) & \mathbb{E}(r_N)\mathbb{E}(r_2) & \ldots & \mathbb{E}(r_N)^2 \end{pmatrix}.$

Now we have $\begin{pmatrix} \mathbb{E}(r_1) \\ \mathbb{E}(r_2) \\ \vdots \\ \mathbb{E}(r_N) \end{pmatrix} = {}^tR \cdot \underline{prob}$ and so $\begin{pmatrix} \mathbb{E}(r_1) \\ \mathbb{E}(r_2) \\ \vdots \\ \mathbb{E}(r_N) \end{pmatrix} \cdot (\mathbb{E}(r_1), \mathbb{E}(r_2), \ldots, \mathbb{E}(r_N)) =$
${}^tR \cdot \underline{prob} \cdot^t ({}^tR \cdot \underline{prob}) = {}^tR \cdot \underline{prob} \cdot {}^t\underline{prob} \cdot R$

Thus we finally end up with the following expression for the covariance matrix:

$C = {}^tR \cdot \mathbb{P} \cdot R - {}^tR \cdot \underline{prob} \cdot {}^t\underline{prob} \cdot R = {}^tR \cdot (\mathbb{P} - {}^t\underline{prob} \cdot \underline{prob}) \cdot R$

Homework problem Consider a model with 5 states: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ with probabilities $P(\omega_1) = \frac{1}{3}, P(\omega_2) = \frac{1}{5}, P(\omega_3) = \frac{1}{5}, P(\omega_4) = \frac{1}{5}, P(\omega_5) = \frac{1}{15}$. Assume there are 7 securities with pay-off matrix

$$D = \begin{pmatrix} 5 & 7 & 0 & 10 & 1 & 6 & 0 \\ 5 & 4 & 0 & 5 & 4 & 6 & -2 \\ 5 & 6 & 0 & 7 & 1 & 6 & 5 \\ 5 & 3 & 0 & 1 & 4 & 6 & 6 \\ 5 & 10 & 3 & 4 & 5 & -1 & -9 \end{pmatrix}$$

Compute the reduced row echelon form of D.

Let the initial price vector $\underline{p}$ be given by $\underline{p} = (4, 10, 1, 6, 2, 5, 3)$.

Show that there are arbitrage strategies.

Adjust the prices of securities 6 and 7 to make the market arbitrage free.

Show (without computing) that the returns of security 1 has $Var = 0$ and that its covariance with any of the other securities is also 0, so security 1 is uncorrelated to any of the other securities.

The reduced row echelon form shows that securities 5 and 6 are redundant i.e. they can be synthesized from securities $1-5$

Compute, for securities $1-5$, the matrix of returns R, the vector of expected returns, the Covariance matrix

$$C = \begin{pmatrix} Var(r_1) & Covar(r_1, r_2) & \dots & Covar(r_1, r_5) \\ Covar(r_2, r_1) & Var(r_2) & \dots & Covar(r_2, r_5) \\ \dots & \dots & \dots & \dots \\ Covar(r_5, r_1) & Covar(r_5, r_2) & \dots & Var(r_5) \end{pmatrix}$$

Compute the correlation matrix of securities $2-5$.